1. [2350/111622 (10 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.

   (a) If vector field $\mathbf{F}(x, y, z)$ possesses continuous second derivatives throughout $\mathbb{R}^3$, then $\nabla \times (\nabla \cdot \mathbf{F}) = \nabla \cdot (\nabla \times \mathbf{F})$.

   (b) The Jacobian, $\dfrac{\partial(x,y)}{\partial(u,v)}$, of the transformation $u = xy, v = xy^2$ is $\dfrac{3}{v}$.

   (c) All gradient vector fields are conservative.

   (d) The curl of $\mathbf{F} = x^2\,\mathbf{i} + (e^z + 2y)\,\mathbf{j} + (xy + yz + xz)\,\mathbf{k}$ lies in the xy-plane.

   (e) The vector field $\mathbf{V} = [2x + \cos(yz)]\,\mathbf{i} + x^2y^2\,\mathbf{j} - 2\left(x^2yz + z\right)\mathbf{k}$ is incompressible.

SOLUTION:

(a) **FALSE** The expression on the left is not defined.

(b) **FALSE**

$$\frac{v}{u} = \frac{xy^2}{xy} = y$$

$$x = \frac{u}{y} = \frac{u}{v/u} = \frac{u^2}{v}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 2u/v & -u^2/v^2 \\ -v/u^2 & 1/u \end{vmatrix} = \left(\frac{2u}{v}\right)\left(\frac{1}{u}\right) - \left(-\frac{v}{u^2}\right)\left(-\frac{u^2}{v^2}\right) = \frac{2}{v} - \frac{1}{v} = \frac{1}{v}$$

(c) **TRUE** Yes, by definition a vector field, $\mathbf{F}$, is called conservative if it is the gradient of some scalar function, f, that is, $\mathbf{F} = \nabla f$

(d) **TRUE**

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & e^z + 2y & xy + yz + xz \end{vmatrix} = (x + z - e^z)\,\mathbf{i} - (y + z)\,\mathbf{j} + 0\,\mathbf{k}$$

(e) **TRUE**

$$\nabla \cdot \mathbf{V} = \frac{\partial}{\partial x}\left[2x + \cos(yz)\right] + \frac{\partial}{\partial y}\left(x^2y^2\right) + \frac{\partial}{\partial z}\left[-2\left(x^2yz + z\right)\right] = 2 + 2x^2y - 2x^2y - 2 = 0$$

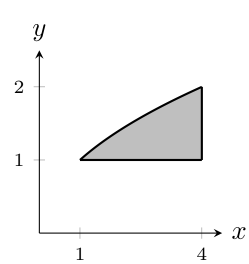
2. [2350/111622 (10 pts)] A thin metal plate (lamina) is placed on the xy-plane. The plate is bounded by $x = 1, x = 4$ and $y = 1, y = \sqrt{x}$, and has a density $\rho(x, y) = 3xy$. Carefully follow the directions below.

   (a) (5 pts) Set up, but **do not evaluate**, an appropriate double integral using the order $\mathrm{d}x\,\mathrm{d}y$ that will give the lamina's moment about the x-axis.

   (b) (5 pts) Set up, but **do not evaluate**, an appropriate double integral using the order $\mathrm{d}y\,\mathrm{d}x$ that will give the lamina's moment about the y-axis.

SOLUTION:

The lamina is as shown in the following figure.



(a)

$$M_x = \iint_{\mathcal{D}} y\rho(x,y)\,\mathrm{d}A = \int_1^2 \int_{y^2}^4 3xy^2\,\mathrm{d}x\,\mathrm{d}y$$

(b)

$$M_y = \iint_{\mathcal{D}} x\rho(x,y)\,\mathrm{d}A = \int_1^4 \int_1^{\sqrt{x}} 3x^2 y\,\mathrm{d}y\,\mathrm{d}x$$

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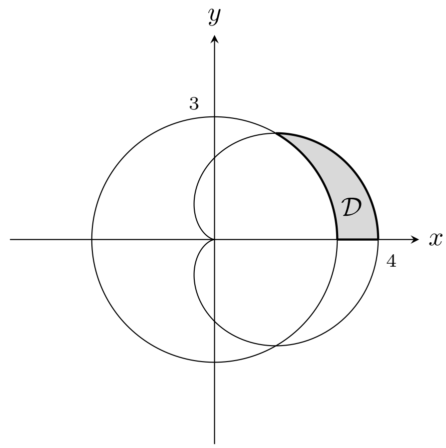
3. [2350/111622 (20 pts)] You have built a new clubhouse whose floor is in the shape of the shaded region $\mathcal{D}$ in the accompanying figure. The curves shown in the figure are $r = 3$ and $r = 2(1+\cos\theta)$. The roof of the clubhouse is given by $f(x,y) = \dfrac{8y}{x^2+y^2}$. Set up and evaluate a double integral to determine the volume of the clubhouse.



SOLUTION:

We need to use polar coordinates. The height of the roof in terms of r and θ is

$$f(r,\theta) = \frac{8r\sin\theta}{r^2} = \frac{8\sin\theta}{r}$$

We also need to find the intersection point of the two curves:

$$3 = 2(1+\cos\theta)$$

$$\frac{1}{2} = \cos\theta \implies \theta = \frac{\pi}{3}$$

$$\text{Volume} = \iint_{\mathcal{D}} f(x,y)\,\mathrm{d}A = \int_0^{\pi/3}\int_3^{2(1+\cos\theta)} \frac{8\sin\theta}{r} r\,\mathrm{d}r\,\mathrm{d}\theta = 8\int_0^{\pi/3}\sin\theta\int_3^{2(1+\cos\theta)}\mathrm{d}r\,\mathrm{d}\theta$$

$$= 8\int_0^{\pi/3}\sin\theta\, r\Big|_3^{2(1+\cos\theta)}\mathrm{d}\theta = 8\int_0^{\pi/3}\sin\theta\,(2\cos\theta - 1)\,\mathrm{d}\theta$$

$$= 8\int_0^{\pi/3}(\sin 2\theta - \sin\theta)\,\mathrm{d}\theta = 8\left(-\frac{1}{2}\cos 2\theta + \cos\theta\right)\Big|_0^{\pi/3}$$

$$= 8\left[\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right) + \frac{1}{2} - \left(-\frac{1}{2}+1\right)\right] = \frac{8}{4} = 2$$

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4. [2350/111622 (17 pts)] A very thin worm is approximated as the curve $\mathcal{C}$ given by $y = \dfrac{1}{3}x^3$ where $1 \le x \le 2$ (x and y are in cm). The worm's density is given by $\delta(x,y) = \dfrac{9y^2}{x^3}$ g/cm. What is the mass of the worm? Include appropriate units in your answer.

SOLUTION:

$$\mathbf{r}(t) = \left\langle t, \frac{1}{3}t^3\right\rangle \implies \|\mathbf{r}'(t)\| = \sqrt{1+t^4}$$

$$\text{mass} = \int_{\mathcal{C}} \frac{9y^2}{x^3}\,\mathrm{d}s = \int_1^2 9\left(\frac{t^6/9}{t^3}\right)\sqrt{1+t^4}\,\mathrm{d}t$$

$$= \int_1^2 t^3\sqrt{1+t^4}\,\mathrm{d}t \overset{u=1+t^4}{=} \frac{1}{4}\int_2^{17} u^{1/2}\,\mathrm{d}u = \frac{1}{6}\left(17^{3/2} - 2^{3/2}\right) = \frac{17\sqrt{17}}{6} - \frac{\sqrt{2}}{3}\text{ grams}$$

5. [2350/111622 (18 pts)] A friend of yours has built a cool art sculpture in the shape of the surface $\frac{1}{2}x^2 - y - \sqrt{3}z = 0$. To protect it from the elements, a protective coating must be applied to that part of the sculpture lying above the triangular region bounded by the lines $x = \sqrt{5}, y = 0$ and $y = 6\sqrt{3}x$ in the xy-plane. The protective coating comes as a liquid in cans, with each can covering 2 square feet. How many cans of the coating must your friend purchase to protect the sculpture? Hint: your answer will be a whole number

SOLUTION:

We need to evaluate a scalar surface integral to determine the area of the surface.

$$g(x, y, z) = \frac{1}{2}x^2 - y - \sqrt{3}z \implies \nabla g = \langle x, -1, -\sqrt{3}\rangle \implies \|\nabla g\| = \sqrt{x^2 + 4}$$

Projecting the surface onto the xy-plane give the region $\mathcal{D}$ of integration as $0 \le x \le \sqrt{5}, 0 \le y \le 6\sqrt{3}x$ with $\mathbf{p} = \mathbf{k}$ and $|\nabla g \cdot \mathbf{p}| = \left|-\sqrt{3}\right| = \sqrt{3}$. Then the area of the surface is

$$\begin{aligned} \text{area} &= \iint_{\mathcal{D}} \frac{\|\nabla g\|}{|\nabla g \cdot \mathbf{p}|}\,\mathrm{d}A = \int_0^{\sqrt{5}} \int_0^{6\sqrt{3}x} \frac{\sqrt{x^2+4}}{\sqrt{3}}\,\mathrm{d}y\,\mathrm{d}x \\ &= \frac{1}{\sqrt{3}} \int_0^{\sqrt{5}} 6\sqrt{3}x\sqrt{x^2+4}\,\mathrm{d}x \qquad (u = x^2 + 4) \\ &= 3\int_4^9 u^{1/2}\,\mathrm{d}u = 2u^{3/2}\Big|_4^9 = 38 \end{aligned}$$

Since each can covers 2 square feet, your friend needs $\left(38 \text{ ft}^2\right)\left(\frac{1 \text{ can}}{2 \text{ ft}^2}\right) = 19$ cans.

6. [2350/111622 (25 pts)] The number of butterflies per cubic foot in a region, $\mathcal{E}$, of space is given by the function $f(x, y, z) = x - z$. The region lies beneath the fourth quadrant $(x > 0, y < 0, z < 0)$ and inside the portion of the sphere $x^2 + y^2 + z^2 = 64$ between the planes $z = -4\sqrt{3}$ and $z = -4$. In parts (b)-(d) set up, **but do not evaluate**, an integral or a sum of integrals that will give the total number of butterflies in the region. For full marks, you must use correct bounds that describe the aforementioned region as stated, not an equivalent one derived from symmetry. Also, be sure to use the standard convention of $0 \le r$, $0 \le \rho$, $0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi$.

(a) (4 pts) Make an appropriately labeled sketch in the rz-plane (constant θ), clearly indicating the region $\mathcal{E}$.

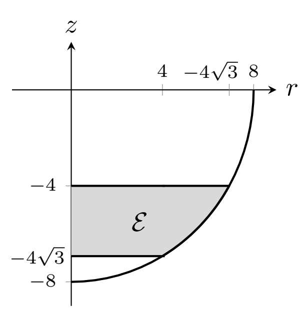
(b) (5 pts) Use cylindrical coordinates and the integration order $\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}\theta$.

(c) (8 pts) Use cylindrical coordinates and the integration order $\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$.

(d) (8 pts) Use spherical coordinates and the integration order $\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$.

SOLUTION:

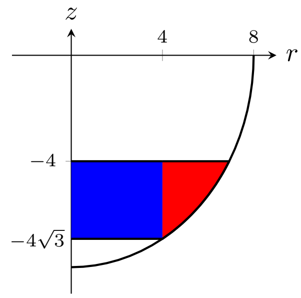
(a) Sketch of the region in the rz-plane (constant θ) is



(b) Using the sketch from part (a)

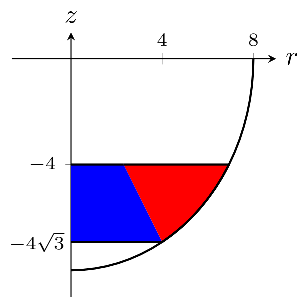
$$\int_{3\pi/2}^{2\pi} \int_{-4\sqrt{3}}^{-4} \int_0^{\sqrt{64-z^2}} (r\cos\theta - z)\, r\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}\theta$$

(c) Sketch of the region in the rz-plane (constant θ) color coded to aid in using cylindrical coordinates



$$\int_{3\pi/2}^{2\pi}\int_{0}^{4}\int_{-4\sqrt{3}}^{-4}(r\cos\theta - z)\,r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta + \int_{3\pi/2}^{2\pi}\int_{4}^{4\sqrt{3}}\int_{-\sqrt{64-r^2}}^{-4}(r\cos\theta - z)\,r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$$

(d) Sketch of the region in the rz-plane (constant θ) color coded to aid in using spherical coordinates



$$\int_{3\pi/2}^{2\pi}\int_{2\pi/3}^{5\pi/6}\int_{-4\sec\phi}^{8}(\rho\sin\phi\cos\theta - \rho\cos\phi)\,\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$$

$$+\int_{3\pi/2}^{2\pi}\int_{5\pi/6}^{\pi}\int_{-4\sec\phi}^{-4\sqrt{3}\sec\phi}(\rho\sin\phi\cos\theta - \rho\cos\phi)\,\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$$

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