1. [2350/101922 (10 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.

(a) The tangent plane to the surface $z = x^2 + 2xy + 2y^2 - 6x + 8y$ at the point $(10, -7)$ is horizontal.

(b) There is no real value of k that makes the function $f(x,y) = \begin{cases} \dfrac{2x^2y}{x^3+y^3} & (x,y) \neq (0,0) \\ k & (x,y) = (0,0) \end{cases}$ continuous on its domain.

(c) If $f(x,y) = e^{x^2+3y}$, $x = \sqrt{2}\cos u \sin 2v$, $y = \sqrt{2}\sin 4u \cos v$, then $\dfrac{\partial f}{\partial u} = 10e$ when $u = v = \dfrac{\pi}{4}$.

(d) The curve in the xy-plane corresponding to all points on the surface $f(x,y) = x^2 - 2x + 4y^2 + 4$ that are 19 units above the xy-plane is a hyperbola.

(e) The instantaneous rate of change of z with respect to y at the point $(1,0,1)$, where $xz^3 + y^2 \ln z + e^x - \cos y + 3xyz = 1$, is 1.

SOLUTION:

(a) **TRUE** $z_x = 2x + 2y - 6 \implies z_x(10,-7) = 20 - 14 - 6 = 0$ and $z_y = 2x + 4y + 8 \implies z_y(10,-7) = 20 - 28 + 8 = 0$

(b) **TRUE** The domain of the function is $\mathbb{R}^2$. For all of $\mathbb{R}^2$ except $(0,0)$, the function is a rational function and therefore continuous. $f(0,0) = k$ so the function is defined at $(0,0)$. However, the limit at $(0,0)$ fails to exist. To see this, approach the origin along the line $y = mx$, where m is a real number. Then

$$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^3+y^3} = \lim_{(x,mx)\to(0,0)} \frac{2x^2(mx)}{x^3+(mx)^3} = \frac{2m}{1+m^3}$$

which is a different number for each value of m, implying the limit does not exist. We thus cannot find a k to make the function continuous at the origin, rendering it not continuous on its domain.

(c) **FALSE** The chain rule gives

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} = e^{x^2+3y}(2x)(-\sqrt{2}\sin u \sin 2v) + e^{x^2+3y}(3)(4)\sqrt{2}\cos 4u \cos v$$

When $u = v = \dfrac{\pi}{4}$,

$$x = \sqrt{2}\cos\left(\frac{\pi}{4}\right)\sin\left(\frac{2\pi}{4}\right) = 1 \quad \text{and} \quad y = \sqrt{2}\sin\left(\frac{4\pi}{4}\right)\cos\left(\frac{\pi}{4}\right) = 0$$

$$\frac{\partial f}{\partial u} = e^{1^2+3(0)}(2)(1)\left(-\sqrt{2}\sin\frac{\pi}{4}\sin\frac{2\pi}{4}\right) + e^{1^2+3(0)}(3)(4)\left(\sqrt{2}\cos\frac{4\pi}{4}\cos\frac{\pi}{4}\right)$$

$$= e(-2\sqrt{2})\left(\frac{\sqrt{2}}{2}\right) + e(12)\sqrt{2}(-1)\left(\frac{\sqrt{2}}{2}\right) = -14e$$

(d) **FALSE** $x^2 - 2x + 4y^2 + 4 = 19$ is equivalent to $(x-1)^2 + 4y^2 = 16$ which is equivalent to $\left(\dfrac{x-1}{4}\right)^2 + \left(\dfrac{y}{2}\right)^2 = 1$ which is an ellipse, a level curve of $f(x,y)$.

(e) **FALSE** $z_y = -\dfrac{F_y}{F_z} = -\dfrac{2y\ln z + \sin y + 3xz}{3xz^2 + \frac{y^2}{z} + 3xy}$ so that $z_y(1,0,1) = -1$

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2. [2350/101922 (21 pts)] The centripetal acceleration (m/s^2) of a particle moving in a circle is $a(r,v) = v^2/r$, where v is the velocity (m/s) and r is the radius (m) of the circle.

(a) (10 pts) Suppose you measure the radius to be roughly 2 m and the velocity to be about 4 m/s with the error in the velocity measurement assumed to be no more than one-half m/s. Use differentials to approximate the maximum error in the measurement of the radius if the error in the acceleration cannot exceed 1 m/s^2.

(b) (11 pts) Find the second order Taylor polynomial of $a(r,v)$ centered at the point $(2,4)$. You need not simplify your answer.

SOLUTION:

(a)

$$\mathrm{d}a = \frac{\partial a}{\partial r}\,\mathrm{d}r + \frac{\partial a}{\partial v}\,\mathrm{d}v = -\frac{v^2}{r^2}\,\mathrm{d}r + \frac{2v}{r}\,\mathrm{d}v$$

Using the given information we want

$$|\mathrm{d}a| = \left| -\frac{4^2}{2^2}\,\mathrm{d}r + \frac{(2)(4)}{2}\left(\frac{1}{2}\right)\right| \le 1$$

$$|-4\,\mathrm{d}r + 2| \le 1$$

$$-1 \le -4\,\mathrm{d}r + 2 \le 1$$

$$-3 \le -4\,\mathrm{d}r \le -1$$

$$\frac{3}{4} \ge \mathrm{d}r \ge \frac{1}{4}$$

(b)

$$a_r = -\frac{v^2}{r^2} \implies a_r(2,4) = -4 \qquad a_v = \frac{2v}{r} \implies a_v(2,4) = 4$$

$$a_{rr} = \frac{2v^2}{r^3} \implies a_{rr} = 4 \qquad a_{vv} = \frac{2}{r} \implies a_{vv}(2,4) = 1$$

$$a_{rv} = -\frac{2v}{r^2} \implies a_{rv}(2,4) = -2 \qquad a(2,4) = 8$$

$$T_2(r,v) = 8 - 4(r-2) + 4(v-4) + \frac{1}{2!}\left[4(r-2)^2 + 2(-2)(r-2)(v-4) + 1(v-4)^2\right]$$

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3. [2350/101922 (32 pts)] You and a friend are hiking in an area whose elevation is described by $z(x,y) = 1000 + x^3 - 3xy - y^3$. When you arrive at the point P given by $(x,y) = (1,-1)$ your friend begins experiencing altitude sickness.

(a) (2 pts) What is the elevation at point P?

(b) (10 pts) What is the slope of the mountain at point P in the direction of $\mathbf{v} = 4\,\mathbf{i} + 3\,\mathbf{j}$?

(c) (10 pts) Your friend knows that to address the altitude sickness she must must get to a lower elevation as quickly as possible. Give a unit vector in the xy-plane showing the direction in which she should begin walking from point P to make this happen.

(d) (10 pts) Prior to your friend getting sick, you were walking along the path through P whose projection in the xy-plane is $\mathbf{r}(t) = \cos(\pi t)\,\mathbf{i} + (t-1)^3\,\mathbf{j}$. What was your instantaneous rate of change of elevation with respect to time at P on this path?

SOLUTION:

(a) $z(1,-1) = 1000 + 1^3 - 3(1)(-1) - (-1)^3 = 1005$

(b) We need the gradient of z and a unit vector $\mathbf{u}$ in the direction of $\mathbf{v}$.

$$z_x = 3x^2 - 3y \implies z_x(1,-1) = 6$$

$$z_y = -3x - 3y^2 \implies z_y(1,-1) = -6$$

$$\nabla z(1,-1) = 6\,\mathbf{i} - 6\,\mathbf{j}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{4\,\mathbf{i} + 3\,\mathbf{j}}{\sqrt{4^2+3^2}} = \frac{4}{5}\,\mathbf{i} + \frac{3}{5}\,\mathbf{j}$$

$$D_{\mathbf{u}}z(1,-1) = \left.\frac{\mathrm{d}z}{\mathrm{d}s}\right|_{(1,-1)} = \nabla z(1,-1)\cdot\mathbf{u} = (6\,\mathbf{i} - 6\,\mathbf{j})\cdot\left(\frac{4}{5}\,\mathbf{i} + \frac{3}{5}\,\mathbf{j}\right) = \frac{6}{5}$$

(c) The vector we seek is $\mathbf{u} = \dfrac{-\nabla z(1,-1)}{\|-\nabla z(1,-1)\|}$.

$$-\nabla z(1,-1) = -6\,\mathbf{i} + 6\,\mathbf{j}$$

$$\|-\nabla z(1,-1)\| = \sqrt{(-6)^2 + 6^2} = 6\sqrt{2}$$

$$\frac{-\nabla z(1,-1)}{\|-\nabla z(1,-1)\|} = \frac{1}{\sqrt{2}}\,(-\mathbf{i} + \mathbf{j})$$

(d) You are at the point P when $t = 0$. We have $\mathbf{r}'(t) = -\pi \sin(\pi t)\,\mathbf{i} + 3(t-1)^2\,\mathbf{j}$.

$$\left.\frac{\mathrm{d}z}{\mathrm{d}t}\right|_{t=0} = \nabla z(1,-1)\cdot \mathbf{r}'(0) = (6\,\mathbf{i} - 6\,\mathbf{j})\cdot \left[-\pi\sin(\pi\cdot 0)\,\mathbf{i} + 3(0-1)^2\,\mathbf{j}\right] = -18$$

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4. [2350/101922 (17 pts)] You have thirty dollars to spend on A pounds of almonds and P pounds of peanuts at Nick and Nancy's Nut Nook. Peanuts cost seventy-five cents a pound and almonds are a dollar and a half per pound. The energy, measured in calories, you get from eating the nuts is given by the function $f(A,P) = 3A^{1/2}P^{3/2}$. Hint: Using fractions of dollars rather than decimals will help simplify the algebra.

   (a) (15 pts) Use Lagrange Multipliers to determine how many pounds of each type of nut you should buy in order to maximize the energy you get from eating them.

   (b) (2 pts) What is the maximum value of energy you can get from eating the nuts? You do not have to simplify your answer.

SOLUTION:

(a) The objective function to be maximized is $f(A,P) = 3A^{1/2}P^{3/2}$ and the constraint is determined by the amount of money we can spend $g(A,P) = \frac{3}{2}A + \frac{3}{4}P = 30$. We have

$$f_A = \frac{3}{2}A^{-1/2}P^{3/2} \quad g_A = \frac{3}{2}$$

$$f_P = \frac{9}{2}A^{1/2}P^{1/2} \quad g_P = \frac{3}{4}$$

Lagrange equations

$$\frac{3}{2}A^{-1/2}P^{3/2} = \frac{3}{2}\lambda \implies \lambda = A^{-1/2}P^{3/2} \tag{1}$$

$$\frac{9}{2}A^{1/2}P^{1/2} = \frac{3}{4}\lambda \implies \lambda = 6A^{1/2}P^{1/2} \tag{2}$$

$$\frac{3}{2}A + \frac{3}{4}P = 30 \tag{3}$$

Equating (1) and (2) and using the result in (3) gives

$$A^{-1/2}P^{3/2} = 6A^{1/2}P^{1/2} \implies P = 6A$$

$$\frac{3}{2}A + \frac{3}{4}(6A) = 30 \implies 6A = 30 \implies A = 5 \implies P = 30$$

You should buy 5 pounds of almonds and 30 pounds of peanuts.

Based on the problem, A and P are nonnegative numbers so the constraint is a closed, bounded set (a line segment in the first quadrant). Consequently, the maximum could occur at the boundary of the set, namely the endpoints of the line segment. However, at these endpoints either A or P is zero giving a value of zero for the objective function, implying that the value at the critical point found above is indeed a maximum.

(b) $f(5,30) = 3\sqrt{5}(30^{3/2}) = (3)30\sqrt{30}\sqrt{5} = (3)(30)(5)\sqrt{6} = 450\sqrt{6}$ calories

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5. [2350/101922 (20 pts)] A gold coin covers the region $x^2 + y^2 \leq 25$ in the xy-plane where the electric charge is given by

$$q(x,y) = x^2 + 2y^2 - x^2y - 3$$

   (a) (15 pts) Are there any relative/local extreme values of charge on the coin? If so, find them. If not, explain why not.

   (b) (5 pts) Without performing any calculations, could the charge on the boundary of the coin be higher and/or lower than any of the values you found in part (a)? Explain briefly.

SOLUTION:

(a) We need to find the critical points of $q(x,y)$ on the disk and apply the Second Derivatives Test to those points.

$$q_x = 2x - 2xy = 2x(1-y) = 0 \implies x = 0, y = 1 \tag{4}$$

$$q_y = 4y - x^2 = 0 \tag{5}$$

If $x = 0$ in (4), then (5) requires $y = 0$ giving $(0,0)$ as a critical point. If $y = 1$ in (4), (5) yields $x = \pm 2$ giving $(2,1)$ and $(-2,1)$ as critical points.

$$q_{xx} = 2 - 2y$$
$$q_{yy} = 4$$
$$q_{xy} = -2x$$
$$D(x,y) = q_{xx}q_{yy} - q_{xy}^2 = 4(2-2y) - 4x^2$$

Applying the second derivatives test at the critical points gives

$$D(0,0) = 4(2-0) - 0 = 8 > 0 \text{ and } q_{yy}(0,0) > 0 \implies q(0,0) = -3 \text{ is a relative minimum}$$
$$D(\pm 2, 1) = 4(0) - 16 < 0 \implies (\pm 2, 1) \text{ are saddle points}$$

The relative minimum charge on the disk is $q(0,0) = -3$. There is no relative maximum charge on the disk.

(b) Yes. $q(x,y)$ is continuous throughout the coin and the coin is a closed, bounded set. The Extreme Value Theorem applies and guarantees that $q(x,y)$ will attain a maximum and minimum on the coin, either at interior critical points or on the boundary. The minimum charge could be less than -3, and the maximum charge will definitely occur on the boundary since there is no relative maximum charge in the interior.

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