1. [2350/092122 (10 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.

   (a) A nonzero vector divided by itself is a unit vector.

   (b) If the dot product of two nonzero vectors is zero, then the vectors are scalar multiples of one another.

   (c) If $\mathbf{a}, \mathbf{u}$ and $\mathbf{v}$ are nonzero vectors and $p \neq 0$ is a scalar, then $\mathbf{a} \cdot (p\mathbf{u} \times \mathbf{v}) = \mathbf{a} \times (\mathbf{u} \cdot p\mathbf{v})$

   (d) $\mathbf{a} = \mathbf{i} + 6\mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = -2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ are orthogonal.

   (e) If the work done to move a crate 4 meters along a horizontal loading dock by applying a force of 5 newtons is 10 joules, then the force was applied at an angle of $\pi/3$ radians to the dock.

   SOLUTION:

   (a) **FALSE** Vectors cannot be divided. A nonzero vector divided by its magnitude is a unit vector.

   (b) **FALSE** The vanishing of the cross product of two nonzero vectors implies the vectors are scalar multiples of one another.

   (c) **FALSE** The quantity on the right is not defined.

   (d) **TRUE** $\mathbf{a} \cdot \mathbf{b} = 0$

   (e) **TRUE** $\theta = \cos^{-1} \dfrac{\mathbf{F} \cdot \mathbf{D}}{\|\mathbf{F}\|\|\mathbf{D}\|} = \cos^{-1} \dfrac{10}{(5)(4)} = \cos^{-1} \dfrac{1}{2} = \dfrac{\pi}{3}$

2. [2350/092122 (14 pts)] The surface of the ocean in a wind-free cove is described by the plane $z = 0$. A seabird sitting on the ocean's surface seeks a meal by following the trajectory (path) given by (distances in meters, times in seconds)

$$\mathbf{r}(t) = \left(\frac{2\sqrt{2}}{3}t^{3/2} + 1\right)\mathbf{i} + \left(\frac{2\sqrt{2}}{3}t^{3/2} + 2\right)\mathbf{j} + \left(\frac{1}{2}t^2 - t\right)\mathbf{k}, \quad t \geq 0$$

   (a) (2 pts) Where is the seabird when it dives into the water?

   (b) (2 pts) Where does the seabird exit the water?

   (c) (10 pts) How far does the seabird travel underwater?

   SOLUTION:

   (a) The seabird's path will intersect the surface of the ocean when the z-component of the path is 0.

$$z = \frac{1}{2}t^2 - t = t\left(\frac{1}{2}t - 1\right) = 0 \implies t = 0, 2 \text{ sec}$$

   The seabird dives into the water when $t = 0$ which places it at $(x, y, z) = (1, 2, 0)$

   (b) The seabird exits the water when $t = 2$, putting it at $(x, y, z) = \left(\frac{11}{3}, \frac{14}{3}, 0\right)$

   (c) The seabird is submerged for $0 \leq t \leq 2$. We need to find the arc length of the path during this time interval.

$$\mathbf{r}'(t) = \left\langle \sqrt{2t}, \sqrt{2t}, t - 1 \right\rangle$$

$$\|\mathbf{r}'(t)\| = \sqrt{\left(\sqrt{2t}\right)^2 + \left(\sqrt{2t}\right)^2 + (t-1)^2} = \sqrt{2t + 2t + t^2 - 2t + 1} = \sqrt{(t+1)^2} = |t+1|$$

   The length of the path underwater is

$$\int_0^2 |t+1| \, \mathrm{d}t \overset{t+1>0}{=} \int_0^2 (t+1) \, \mathrm{d}t = \left(\frac{1}{2}t^2 + t\right)\Bigg|_0^2 = 4$$

   The seabird's underwater path is 4 meters long.

3. [2350/092122 (26 pts)] A laser situated on the surface $x^2 - y^2 + z^2 + 2 = 0$ at the point $P(-1, 2, 1)$ is aimed at the point $Q(1, -1, 2)$.

   (a) (4 pts) Name the surface.

(b) (4 pts) Describe the intersection of the surface with the plane $y = \sqrt{2}$.

(c) (8 pts) Find a parameterization of the laser beam (assumed to be a straight line). Be sure to include in your answer appropriate values of the parameter that provide the correct orientation of the beam.

(d) (10 pts) The laser beam hits the surface at another point R.

i. (5 pts) Find the position vector of R.

ii. (5 pts) How far is it from the laser to R?

SOLUTION:

(a) Hyperboloid of 2 sheets.

(b) If $y = \sqrt{2}$ in the equation of the surface, we have $x^2 - (\sqrt{2})^2 + z^2 + 2 = x^2 + z^2 = 0$ which is the single point $(0, \sqrt{2}, 0)$.

(c) A vector in the direction of the beam is $\overrightarrow{PQ} = \langle 1, -1, 2\rangle - \langle -1, 2, 1\rangle = \langle 2, -3, 1\rangle$. The most straightforward parameterization of the beam is

$$x(t) = -1 + 2t, \quad y(t) = 2 - 3t, \quad z(t) = 1 + t, \quad t \geq 0$$

with another possibility being

$$x(t) = 1 + 2t, \quad y(t) = -1 - 3t, \quad z(t) = 2 + t, \quad t \geq -1$$

(d) i. Using the parameterization of the line in the equation of the surface yields

$$\begin{aligned}(-1+2t)^2 - (2-3t)^2 + (1+t)^2 + 2 &= 1 - 4t + 4t^2 - (4 - 12t + 9t^2) + 1 + 2t + t^2 + 2 \\ &= 10t - 4t^2 \\ &= 2t(5-2t) = 0 \\ \implies t &= 0, \frac{5}{2}\end{aligned}$$

The position vector of the point of intersection is

$$\left\langle x\left(\frac{5}{2}\right), y\left(\frac{5}{2}\right), z\left(\frac{5}{2}\right)\right\rangle = \left\langle 4, -\frac{11}{2}, \frac{7}{2}\right\rangle$$

Using the other parameterization above yields $t = -1, \dfrac{3}{2}$.

ii. Use the distance formula

$$d = \sqrt{(4+1)^2 + \left(-\frac{11}{2} - 2\right)^2 + \left(\frac{7}{2} - 1\right)^2} = \sqrt{5^2 + \left(-\frac{15}{2}\right)^2 + \left(\frac{5}{2}\right)^2} = \sqrt{\frac{350}{4}} = \frac{5}{2}\sqrt{14}$$

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4. [2350/092122 (15 pts)] The unit normal vector to the curve $\mathbf{r}(t) = \left\langle \frac{1}{3}\sin 3t, \sqrt{3}\,t, -\frac{1}{3}\cos 3t\right\rangle$ is $\mathbf{N}(t) = \langle -\sin 3t, 0, \cos 3t\rangle$. Find the equation of the osculating plane when $t = \dfrac{\pi}{3}$. Write your answer in the form $ax + by + cz = d$.

SOLUTION:

We need a point in the plane and its normal. A point in the plane is $\mathbf{r}\left(\frac{\pi}{3}\right) = \left(0, \frac{\pi\sqrt{3}}{3}, \frac{1}{3}\right)$. The normal to the osculating plane is the binormal, $\mathbf{B}(t)$, to the curve.

$$\mathbf{r}'(t) = \langle \cos 3t, \sqrt{3}, \sin 3t\rangle$$

$$\|\mathbf{r}'(t)\| = \sqrt{\cos^2 3t + 3 + \sin^2 3t} = 2$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{2}\left\langle \cos 3t, \sqrt{3}, \sin 3t\right\rangle$$

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{2}\cos 3t & \frac{\sqrt{3}}{2} & \frac{1}{2}\sin 3t \\ -\sin 3t & 0 & \cos 3t \end{vmatrix} = \left\langle \frac{\sqrt{3}}{2}\cos 3t, -\frac{1}{2}, \frac{\sqrt{3}}{2}\sin 3t\right\rangle$$

When $t = \dfrac{\pi}{3}$, $\mathbf{B}\left(\dfrac{\pi}{3}\right) = \left\langle -\dfrac{\sqrt{3}}{2}, -\dfrac{1}{2}, 0 \right\rangle$ giving

$$-\frac{\sqrt{3}}{2}(x-0) - \frac{1}{2}\left(y - \frac{\pi\sqrt{3}}{3}\right) + 0\left(z - \frac{1}{3}\right) = 0$$

$$\sqrt{3}x + y = \frac{\pi\sqrt{3}}{3}$$

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5. [2350/092122 (35 pts)] A dragonfly is buzzing along a path with a velocity of $\mathbf{v}(t) = \mathbf{i} + 2(t-1)\,\mathbf{j} + 0\,\mathbf{k}$, where t is a real number.

   (a) (10 pts) If the dragonfly is at the point $(3, 0, 3)$ when $t = 2$, where is it when $t = 4$?

   (b) (10 pts) Compute the tangential component of the acceleration of the dragonfly's path. Can this be used to determine if there are points on the path where the dragonfly's speed is not changing? If so, find them. If not, explain why not.

   (c) (10 pts) Compute the normal component of the acceleration of the dragonfly's path. Can this be used to determine if there are points on the path where the dragonfly's direction is not changing? If so, find them. If not, explain why not.

   (d) (5 pts) Does the dragonfly's path approach a straight line as $t \to \infty$? Justify your answer.

SOLUTION:

(a)

$$\mathbf{r}(t) = \int \mathbf{v}(t)\,\mathrm{d}t = \int \left[\mathbf{i} + 2(t-1)\,\mathbf{j} + 0\,\mathbf{k}\right] \mathrm{d}t = (t + c_1)\,\mathbf{i} + \left[(t-1)^2 + c_2\right]\,\mathbf{j} + c_3\,\mathbf{k}$$

$$\mathrm{r}(0) = 3\,\mathbf{i} + 0\,\mathbf{j} + 3\,\mathbf{k} = (2 + c_1)\,\mathbf{i} + (1 + c_2)\,\mathbf{j} + c_3\,\mathbf{k} \implies c_1 = 1, c_2 = -1, c_3 = 3$$

$$\mathbf{r}(t) = (t+1)\,\mathbf{i} + (t^2 - 2t)\,\mathbf{j} + 3\,\mathbf{k}$$

$$\mathbf{r}(4) = 5\,\mathbf{i} + 8\,\mathbf{j} + 3\,\mathbf{k}$$

(b) Speed changes are determined by the tangential acceleration.

$$a_T = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{\|\mathbf{r}'(t)\|} = \frac{\mathbf{v}(t) \cdot \mathbf{v}'(t)}{\|\mathbf{v}(t)\|} = \frac{\left[\mathbf{i} + 2(t-1)\,\mathbf{j} + 0\,\mathbf{k}\right] \cdot 2\,\mathbf{j}}{\sqrt{1 + [2(t-1)]^2}} = \frac{4(t-1)}{\sqrt{4t^2 - 8t + 5}}$$

This will vanish, implying that the dragonfly's speed is not changing, when $t = 1$ which is at $\mathbf{r}(1) = 2\,\mathbf{i} - \mathbf{j} + 3\,\mathbf{k}$

(c) Direction changes are determined by the normal acceleration.

$$a_N = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\|\mathbf{v}(t) \times \mathbf{v}'(t)\|}{\|\mathbf{v}(t)\|} = \frac{\|\left[\mathbf{i} + 2(t-1)\,\mathbf{j} + 0\,\mathbf{k}\right] \times 2\,\mathbf{j}\|}{\sqrt{1 + [2(t-1)]^2}} = \frac{\|2\,\mathbf{k}\|}{\sqrt{4t^2 - 8t + 5}} = \frac{2}{\sqrt{4t^2 - 8t + 5}}$$

Since a_N is never zero, the dragonfly's direction is always changing.

(d)

$$\kappa(t) = \frac{\|\mathbf{v}(t) \times \mathbf{v}'(t)\|}{\|\mathbf{v}(t)\|^3} = \frac{2}{(4t^2 - 8t + 5)^{3/2}}$$

$$\lim_{t \to \infty} \kappa(t) = 0$$

This says that the dragonfly's path gets close to a straight line that has no curvature as t grows without bound.

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