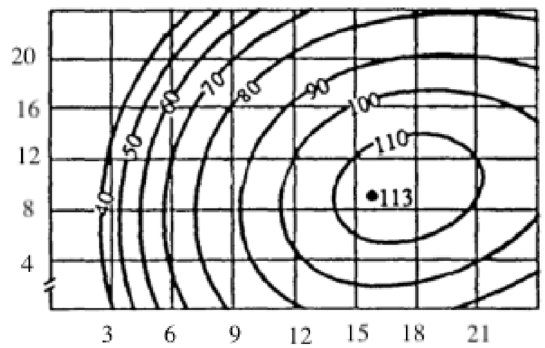
APPM 2350—Final Exam

Saturday, Dec 11th 7:30am-10am 2021

Show all your work and simplify your answers. Answers with no justification will receive no points. You are allowed one 8.5×11-in page of notes (TWO sided). You may NOT use a calculator, smartphone, smartwatch, the Internet or any other electronic device.

Problem 1 (35 points)
Use the portion of the level curve graph of $g(x, y)$ shown below to answer the following questions. You can assume $g(x, y)$ is a continuous function with continuous partial derivatives.



Do NOT try to find an expression for the actual function $g(x, y)$ (no credit will be given). Use ONLY the information provided on the level curve plot to answer the following questions:

(a) Estimate $g_y(9, 16)$. For credit, *show work* justifying your estimate.

(b) Estimate the derivative of $g(x, y)$ at the point $(9, 16)$ in the direction $-3\mathbf{i}+4\mathbf{j}$. For credit, *show work* justifying/explaining your estimate.

(c) Estimate $\mathbf{n}$, a unit vector that is normal to the surface $z = g(x, y)$ at the point $(9, 16, 80)$. For credit, *show work* justifying your estimate.

(d) Based on the level curves shown, are there any points in the domain where $\nabla g = \overrightarrow{0}$? If not, explain why. If so, give the value of $g(x, y)$ at the point(s).

(e) Let $\mathcal{C}$ be the level curve defined by $g(x, y) = 110$ and oriented counter-clockwise. Determine whether the work done by $\mathbf{G} = \nabla g$ around $\mathcal{C}$ is positive, zero or negative. Justify your answer.

(f) Let $\mathcal{C}$ be the level curve defined by $g(x, y) = 110$. Determine whether the total outward flux of $\mathbf{G} = \nabla g$ through $\mathcal{C}$ is positive, zero or negative. Justify your answer.

(g) **Evaluate** $\int_{\mathcal{C}_1} \nabla g \cdot \mathbf{dr}$, where $\mathcal{C}_1$ is the straight line path from $(x, y) = (21, 12)$ to $(x, y) = (9, 20)$.

SOLUTION:

(a) $g_y \approx \frac{\Delta g}{\Delta y}$

There are multiple acceptable answers, as long as you justify your reasoning.

Option 1: Using a forward difference at the point $(9, 16)$:

$$g_y(9, 16) \approx \frac{g(9, 20) - g(9, 16)}{20 - 16} = \frac{70 - 80}{4} = -\frac{5}{2}$$

Option 2: Based on the level curves shown we can approximate $g(9, 12) \approx 85$ (or any other number between 85 and 90) and use a centered difference:

$$g_y(9, 16) \approx \frac{g(9, 20) - g(9, 12)}{20 - 12} = \frac{70 - 85}{8} = -\frac{15}{8}$$

(b)

$$D_u g = \frac{dg}{ds} = \nabla g(9,16)\cdot \mathbf{u} \text{ where } u = \frac{-3\mathbf{i}+4\mathbf{j}}{\sqrt{3^2+4^2}} = -\frac{3}{5}\mathbf{i}+\frac{4}{5}\mathbf{j}$$

There are multiple acceptable ways to approximate this, as long as you justify your reasoning.
Option 1: Approximate $\nabla g(9,16)$ using forward differences:
From part (a) we have $g_y \approx -\frac{5}{2}$
Using a forward difference at the point $(9,16)$:

$$g_x(9,16) \approx \frac{g(12,16)-g(9,16)}{12-9} = \frac{90-80}{3} = \frac{10}{3}$$

Thus using forward differences, $\nabla g(9,16) \approx \langle \frac{10}{3}, -\frac{5}{2}\rangle$
Thus

$$D_u g = \nabla g(9,16)\cdot \mathbf{u} \approx \langle \frac{10}{3}, -\frac{5}{2}\rangle \cdot \langle -\frac{3}{5}, \frac{4}{5}\rangle = -2-2 = -4$$

Option 2: Approximate $\frac{dg}{ds}$ in the direction $\mathbf{u}$ directly on the level curve plot using forward differences. Starting at the point $(9,16)$ and moving in the direction $\langle -3,4\rangle$ (i.e. a distance of 5 diagonally across one of the boxes) we see:

$$\frac{dg}{ds} \approx \frac{\Delta g}{\Delta s} = \frac{g(6,20)-g(9,16)}{5} = \frac{50-80}{5} = -6$$

Note that a more accurate forward approximation would use just the closest level curve in the direction $\langle -3,4\rangle$, i.e.:

$$\frac{dg}{ds} \approx \frac{\Delta g}{\Delta s} = \frac{70-80}{2} = -5$$

Other approximations (including centered differences of the above calculations) are also acceptable as long as you show work justifying your steps.

(c) Let

$$f(x,y,z) = g(x,y) - z$$

The surface $z = g(x,y)$ is one level surface of $f(x,y,z)$, thus ∇f is normal to this surface.

$$\nabla f(x,y,z) = \langle g_x(x,y), g_y(x,y), -1\rangle$$

Thus

$$\mathbf{n} = \frac{\nabla f(9,16,80)}{||\nabla f(9,16,80)||}$$

is normal to the graph of the surface at the point $(9,16,80)$.
Using our estimates from part (a) and (b):

$$\nabla f(9,16,80) = \langle g_x(9,16), g_y(9,16), -1\rangle \approx \langle \frac{10}{3}, -\frac{5}{2}, -1\rangle$$

Thus

$$\mathbf{n} \approx \frac{\langle \frac{10}{3}, -\frac{5}{2}, -1\rangle}{\sqrt{(10/3)^2+(-5/2)^2+1}}$$

(d) Yes, there is a critical point when g(x,y) =113

(e) Since ∇g is a conservative field, the work done around any closed curve is zero. Since $g(x,y) = 110$ is a closed curve, the work done is zero.

(f) The total flux is $\oint_C \nabla g \cdot \mathbf{n}\, ds$ where $\mathbf{n}$ is the outward pointing unit normal. Since ∇g points in the direction of greatest increase, for this function it points inward to the level curves and thus $\mathbf{n} = -\frac{\nabla g}{||\nabla g||}$ is the outward unit normal to the level curve $g(x,y) = 110$. Thus the total flux is

$$\oint_C \nabla g \cdot \mathbf{n}\, ds = \oint_C \nabla g \cdot \left(\frac{-\nabla g}{||\nabla g||}\right) ds = -\oint_C ||\nabla g||\, ds < 0$$

(g) We can use the Fundamental Theorem of Line Integrals:

$\int_C \nabla g \cdot d\mathbf{r} = g(9,20) - g(21,12) = 70 - 110 = \boxed{-40}$

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Problem 2 (30 points)

The following questions are not related:

(a) Given

$$\mathbf{F} = z\,\mathbf{i} + x\,\mathbf{j} - 3e^{y^2}\arctan(z^2)\,\mathbf{k}$$

Find the outward flux through the surface that consists of the part of the cylinder $x^2 + y^2 = 16$ that lies in the **first octant** between $z = 0$ and $z = 5$. (By outward we mean oriented away from the z-axis. Note that this surface is not closed).

(b) Evaluate

$$\int_{\mathcal{C}} (2x\cos y + 3)\,dx - (x^2 \sin y + 2y)\,dy$$

where $\mathcal{C}$ is the path $\mathbf{r}(t) = \cos^3 t\mathbf{i} + \sin^3 t\mathbf{j}, \quad 0 \le t \le \pi/2.$

<u>**SOLUTION:**</u>

(a) We need to compute $\iint_{\mathcal{S}} \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S$, a standard vector surface (flux) integral.

$$g(x,y,z) = x^2 + y^2 \implies \nabla g = \langle 2x, 2y, 0\rangle \quad \text{(use positive gradient for outward normal)}$$

$$\mathbf{F}\cdot\nabla g = \langle z, x, -3e^{y^2}\arctan(z^2)\rangle \cdot \langle 2x, 2y, 0\rangle = 2x(y+z)$$

project onto the yz-plane $\implies$ $\mathbf{p} = \mathbf{i}$, $|\nabla g\cdot\mathbf{p}| = |2x| = 2x$ since $x > 0$ and

$$\mathcal{R} = \left\{(x,y,z) \in \mathbb{R}^3 \,\middle|\, x = 0, 0 \le y \le 4, 0 \le z \le 5\right\}$$

$$\iint_{\mathcal{S}} \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S = \iint_{\mathcal{R}} \mathbf{F}\cdot\frac{\nabla g}{|\nabla g\cdot\mathbf{p}|}\,\mathrm{d}A = \boxed{\int_0^5\int_0^4 (y+z)\,\mathrm{d}y\,\mathrm{d}z = 90}$$

Note: Projecting onto the xz-plane results in an integrand of $\dfrac{x(y+z)}{y}$ which, using the surface to eliminate y, gives

$$\boxed{\int_0^5\int_0^4 \left(\frac{xz}{\sqrt{16-x^2}} + x\right)\mathrm{d}x\,\mathrm{d}z = 90}$$

(b) Direct integration is not possible as the integrand will contain terms like $\cos(\sin^3 t)$. Noting that the integral can be written alternatively as $\int_{\mathcal{C}} \mathbf{F}\cdot\mathrm{d}\mathbf{r}$ with $\mathbf{F} = \big\langle 2x\cos y + 3, -x^2\sin y - 2y\big\rangle = \langle P, Q\rangle$, we check to see if $\mathbf{F}$ is conservative.

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -2x\sin y - (-2x\sin y) = 0$$

Since $\mathbf{F}$ is defined throughout $\mathbb{R}^2$, which is simply connected, and $\nabla\times\mathbf{F} = \mathbf{0}$, $\mathbf{F}$ is conservative. This means that the integral is path independent and also that a potential function, f, exists such that $\mathbf{F} = \nabla f$. This provides two options:

Option 1 - find the potential and use the Fundamental Theorem for Line Integrals

$$\frac{\partial f}{\partial x} = 2x\cos y + 3 \implies f(x,y) = x^2\cos y + 3x + g(y)$$

$$\frac{\partial f}{\partial y} = -x^2\sin y + g'(y) = -x\sin y - 2y \implies g'(y) = -2y \implies g(y) = -y^2 + c$$

$$f(x,y) = x^2\cos y + 3x - y^2 + c$$

Then we have

$$\int_{\mathcal{C}} (2x\cos y + 3)\,\mathrm{d}x - (x^2\sin y + 2y)\,\mathrm{d}y = \int_{\mathcal{C}} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_{\mathcal{C}} \nabla f \cdot \mathrm{d}\mathbf{r}$$

$$\boxed{= f(0,1) - f(1,0) = (-1+c) - (4+c) = -5}$$

Option 2 - integrate along another path

To avoid more complex integrations which would occur if using the line segment between $(1,0)$ and $(0,1)$, we choose the path $(1,0) \to (0,0) \to (0,1)$ consisting of two line segments. For the part along the x-axis we have $x = 1-t, y = 0, 0 \le t \le 1$ with $\mathrm{d}x = -\mathrm{d}t$ and $\mathrm{d}y = 0$ and for the part along the y-axis we use $x = 0, y = t, 0 \le t \le 1$ with $\mathrm{d}x = 0$ and $\mathrm{d}y = \mathrm{d}t$. Then

$$\int_{\mathcal{C}} (2x\cos y + 3)\,\mathrm{d}x - (x^2\sin y + 2y)\,\mathrm{d}y = \int_0^1 [2(1-t)+3]\,(-\mathrm{d}t) + \int_0^1 -2t\,\mathrm{d}t$$

$$= \int_0^1 (-5+2t)\,\mathrm{d}t + \int_0^1 -2t\,\mathrm{d}t = \int_0^1 -5\,\mathrm{d}t = -5$$

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Problem 3 (25 points)
A metal plate lies in the xy-plane in the region $0 \le x \le 4 - y^2$
(i.e the region in quadrants I and IV bounded by the y-axis and the curve $x = 4 - y^2$).

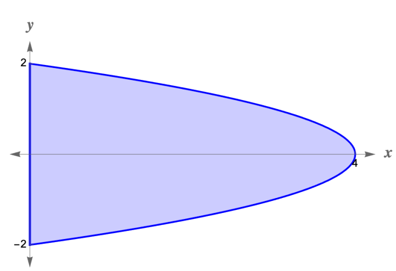
(a) Sketch and shade the region where the plate lies. Label any intercepts.
(b) Find the work done by

$$\mathbf{G} = \left\langle yx^2 - 3y + e^{\cos(x)}, \quad \frac{x^3}{3} - \arctan y \right\rangle$$

counterclockwise once around the boundary of the plate.
(c) Let the temperature at any point on the plate be given by $T(x,y) = y^2 + x^2 - x$. Find the location(s) of the hottest and coldest points on the plate. Show work fully justifying your answer.

<u>**SOLUTION:**</u>

(a) The region looks like this:



(b) The boundary curve C of the region is piecewise smooth and closed, and the vector field $\mathbf{G}$ has continuous first partial derivatives, so we should use Green's Theorem. With $\mathbf{G} = \langle P(x,y), Q(x,y)\rangle$,

$$\begin{aligned}
\oint_C \mathbf{G}\cdot d\mathbf{r} &= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \\
&= \iint_R [x^2 - (x^2 - 3)]\, dA \\
&= \iint_R 3\, dA \\
&= 3\int_{-2}^{2}\int_0^{4-y^2} 1\, dx\, dy \\
&= 3\int_{-2}^{2} \Big[x\Big]_0^{4-y^2} dy \\
&= 3\int_{-2}^{2} (4 - y^2)\, dy \\
&= 3\left[4y - \frac{y^3}{3}\right]_{-2}^{2} \\
&= 3\left\{\left[4(2) - \frac{(2)^3}{3}\right] - \left[4(-2) - \frac{(-2)^3}{3}\right]\right\} \\
&= 3\left[\left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right)\right] \\
&= 24 - 8 + 24 - 8 \\
&= \boxed{32}
\end{aligned}$$

(c) By the Extreme Value Theorem there must exist absolute max and min temperatures on the plate. We have to optimize the temperature function $T(x,y) = y^2 + x^2 - x$ *inside* the plate and on the *boundary* of the plate.

For full credit you must find all possible max/min on the interior and boundary, and compare the temperatures at each point to find the absolute max and min.

To optimize $T(x,y)$ on the inside of the plate, note that $\nabla T(x,y) = \langle 2x - 1, 2y\rangle$, which equals $\langle 0, 0\rangle$ when

$$2x - 1 = 0 \implies x = \frac{1}{2} \quad \text{and} \quad 2y = 0 \implies y = 0$$

Since $\nabla T(x,y)$ always exists (T is a polynomial in two variables), $\left(\frac{1}{2}, 0\right)$ is the only critical point of $T(x,y)$, which is in the interior of the metal plate. We will evaluate $T(x,y)$ at this point after we have found all points of interest.

To optimize $T(x,y)$ on the boundary of the plate, we break the curve C into two pieces, say C_1 is the line segment on the y-axis given by $x = 0$ and $-2 \le y \le 2$, and C_2 is the curve $x = 4 - y^2$ for $-2 \le y \le 2$.

On C_1, since $x = 0$ we study the function $T_1(y) = T(0,y) = y^2$ on the interval $[-2, 2]$. Since $T_1'(y) = 2y$ which equals zero when $y = 0$, we have three points of interest (when $y = 0$ and including the endpoints $y = \pm 2$) to consider later:

$$(0,0) \qquad (0,2) \qquad (0,-2)$$

To optimize $T(x,y)$ on the curve C_2 there are two methods we could use:

Method 1: just like on C_1, since $x = 4 - y^2$ on C_2 we study the function

$$T_2(y) = T(4 - y^2, y) = y^2 + (4 - y^2)^2 - (4 - y^2) = y^2 + 16 - 8y^2 + y^4 - 4 + y^2 = y^4 - 6y^2 + 12$$

for $-2 \le y \le 2$.

But

$$T_2'(y) = 4y^3 - 12y = 4y(y^2 - 3)$$

which equals zero when $y = 0, \pm\sqrt{3}$. This gives us three more points of interest to consider:

$$(4, 0) \qquad (1, \sqrt{3}) \qquad (1, -\sqrt{3})$$

(note that the 'endpoint' $y = \pm 2$ are the same as on C_1)

Method 2: we could use Lagrange multipliers. Let $g(x, y) = x + y^2$. Then the curve C_2 is the level curve $g(x, y) = 4$. We want to solve the system

$$\nabla T(x, y) = \lambda \nabla g(x, y) \quad \text{and} \quad g(x, y) = 4$$

or

$$\langle 2x - 1, 2y\rangle = \lambda\langle 1, 2y\rangle \qquad \text{and} \qquad x = 4 - y^2$$

or

$$\begin{aligned} 2x - 1 &= \lambda \\ 2y &= 2\lambda y \\ x &= 4 - y^2 \end{aligned}$$

The second equation tells us that

$$2y - 2\lambda y = 0 \quad \text{or} \quad 2y(1 - \lambda) = 0,$$

so either $y = 0$ or $\lambda = 1$. If $y = 0$, the third equation gives $x = 4$, so $(4, 0)$ is a point of interest.

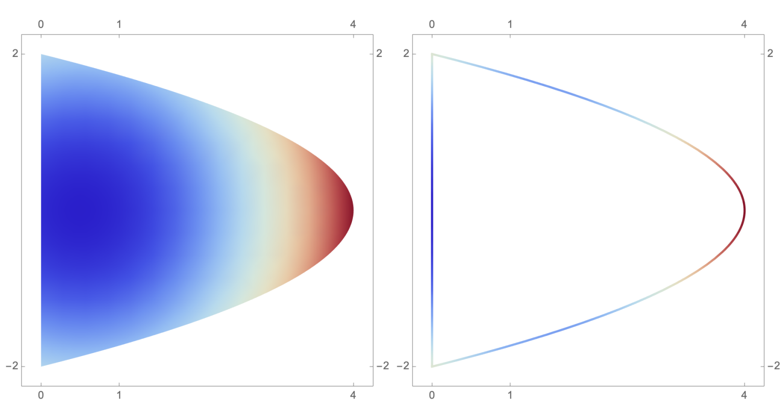
If $\lambda = 1$, then the first equation says that $2x - 1 = 1$ or $x = 1$. The third equation then gives $y = \pm\sqrt{3}$. We get the two points of interest $(1, \pm\sqrt{3})$.

To finish the problem, we evaluate $T(x, y)$ at all of the points of interest:

$$\begin{aligned} T(\tfrac{1}{2}, 0) &= -\tfrac{1}{4} \\ T(0, 0) &= 0 \\ T(0, -2) &= 4 \\ T(0, 2) &= 4 \\ T(4, 0) &= 12 \\ T(1, \sqrt{3}) &= 3 \\ T(1, -\sqrt{3}) &= 3 \end{aligned}$$

The hottest point on the plate is therefore(4,0) and the coldest point is ($\frac{1}{2}, 0$).

For fun, this is what the temperature distribution on the inside and on the boundary of the plate looks like:



Problem 4 (35 points)

The integral

$$\int_{0}^{2\sqrt{2}} \int_{-\sqrt{8-y^2}}^{\sqrt{8-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{16-x^2-y^2}} dz\, dx\, dy$$

represents the volume of a 3D region, $\mathcal{E}$.

(a) Sketch and shade a 2D cross section of the object in the rz-plane. Label the (r, z) coordinates on all corners of the cross section.

(b) Sketch and shade the projection of the object onto the xy-plane. Label any intercepts.

(c) Set up but DO NOT EVALUATE, equivalent integral(s) to find the volume of the object using:

(i) Cylindrical coordinates in the order $dz\, dr\, d\theta$

(ii) Spherical coordinates in the order $d\rho\, d\phi\, d\theta$

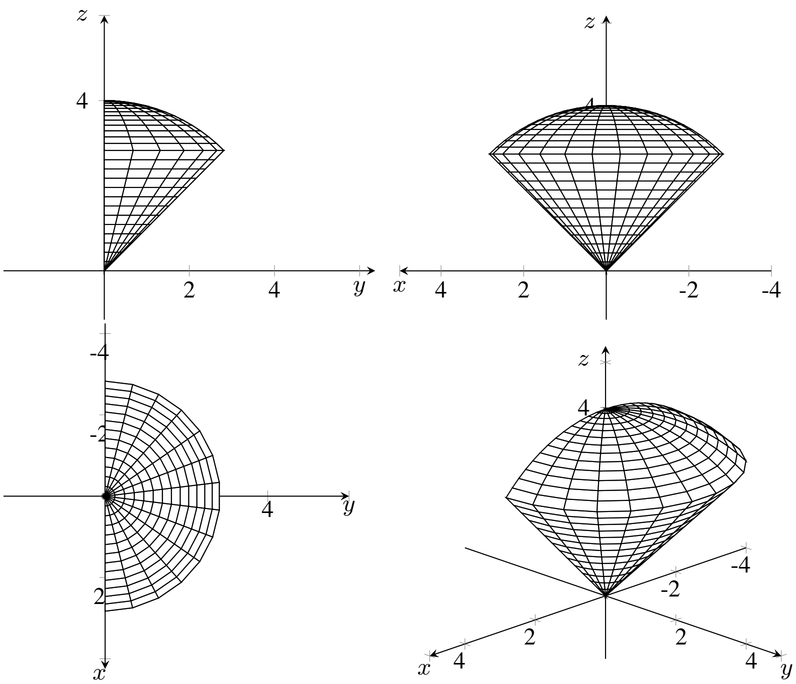
(d) Calculate the total outward flux of

$$\boldsymbol{F} = (xz^2 + \tan(y^2 z))\mathbf{i} + (yx^2 - xe^{cos(z)})\mathbf{j} + (zy^2)\mathbf{k}$$

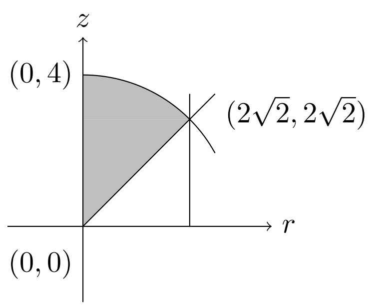
across the entire surface of $\mathcal{E}$.

<u>**SOLUTION:**</u>

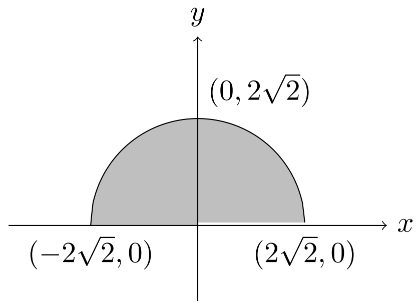
The surface looks like the figure below.



(a)



(b)



(c)

(i) $\boxed{\int_0^{\pi}\int_0^{2\sqrt{2}}\int_r^{\sqrt{16-r^2}} r\,dz\,dr\,d\theta}$

(ii) $\boxed{\int_0^{\pi}\int_0^{\frac{\pi}{4}}\int_0^{4} \rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta}$

(d) Calculating

$$\begin{aligned}\int_V (\nabla\cdot\boldsymbol{F})\,dV &= \int_V \left(z^2+x^2+y^2\right)dV = \int_0^{\pi}\int_0^{\frac{\pi}{4}}\int_0^{4}\left(\rho^2\right)\rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta\\ &= \left(\int_0^{\pi} d\theta\right)\left(\int_0^{\frac{\pi}{4}}\sin(\phi)\,d\phi\right)\left(\int_0^{4}\rho^4\,d\rho\right) = \left(\theta\Big|_0^{\pi}\right)\left(-\cos(\phi)\Big|_0^{\frac{\pi}{4}}\right)\left(\frac{\rho^5}{5}\Big|_0^{4}\right)\\ &= \pi\left(\cos(0)-\cos\left(\frac{\pi}{4}\right)\right)\frac{4^5}{5} = \pi\left(1-\frac{\sqrt{2}}{2}\right)\frac{1024}{5},\end{aligned}$$

by the Divergence Theorem, the outward flux is

$$\oiint \boldsymbol{F}\cdot\boldsymbol{n}\,dS = \int_V (\nabla\cdot\boldsymbol{F})\,dV = \boxed{\pi\left(1-\frac{\sqrt{2}}{2}\right)\frac{1024}{5}.}$$

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Problem 5 (25 points)
Let $\mathcal{C}$ be the path parameterized by

$$\mathbf{r}(t) = \langle 2\cos t,\ \ 2\sin t,\ \ 3-\cos t\rangle,\quad 0\le t\le 2\pi$$

.

(a) $\mathcal{C}$ lies in a plane. Find the equation of that plane.

(b) Let

$\mathbf{F}(x,y,z) = \left\langle e^{x^2}+2z^2,\ \ \sinh(y^2),\ \ \cos(z^2)+zy\right\rangle$

Find the circulation of $\mathbf{F}(x,y,z)$ around $\mathcal{C}$.

SOLUTION:

(a) To find the equation of the plane we need a point and a normal vector.

One possible point : $t=0 \implies (x,y,z)=(2,0,2)$.

To find the normal vector:

Option 1: Find 3 points on the curve, create 2 vectors between them and find their cross product:

P_1: $t=0 \implies (x,y,z)=(2,0,2)$.

P_2: $t=\pi/2 \implies (x,y,z)=(0,2,3)$.

P_3: $t=\pi \implies (x,y,z)=(-2,0,4)$.

$$P_1P_2 \times P_1P_3 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 2 & 1 \\ -4 & 0 & 2 \end{vmatrix} = \langle 4, 0, 8 \rangle$$

Thus the plane is $4(x-2) + 0(y-0) + 8(z-2) = 0$

$$\boxed{\implies x + 2z = 6}$$

Option 2:

The normal vector to the osculating plane (binormal vector of the curve) will point in the direction of $\mathbf{r}'(t) \times \mathbf{r}''(t)$:

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2\sin(t) & 2\cos(t) & \sin(t) \\ -2\cos(t) & -2\sin(t) & \cos(t) \end{vmatrix}$$

$= \left\langle 2\cos^2(t) + 2\sin^2(t), 2\sin(t)\cos(t) - 2\sin(t)\cos(t), 4\sin^2(t) + 4\cos(t) \right\rangle$
$= \langle 2, 0, 4 \rangle$

Thus, the osculating plane to the curve $\mathcal{C}$ is constant i.e. $\mathcal{C}$ lies in a plane which has a normal vector that points in the direction of $\langle 2, 0, 4 \rangle$.

Thus the plane is $2(x-2) + 0(y-0) + 4(z-2) = 0$

$$\boxed{\implies x + 2z = 6}$$

(b) Note that integrating $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt$ would be impossible. We can start by checking to see if this vector field is conservative:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{x^2} + 2z^2 & \sinh(y^2) & \cos(z^2) + zy \end{vmatrix} = \langle z, 4z, 0 \rangle$$

Thus $\mathbf{F}$ is not conservative.

However, since C is a closed curve, Stokes' theorem tells us that

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int\int_{\mathcal{S}} (\nabla \times \mathbf{F}) \cdot \hat{n}\, dS$$

If we model the plane by the equation $x + 2z = d$ it becomes a level surface of the function $g(x, y, z) = x + 2z$.

Thus $\nabla g(x, y, z) = \langle 1, 0, 2 \rangle$. Checking orientation, this follows the right-hand rule with the direction of C.

Projecting onto the $xy-$plane we get:

$$(\nabla \times \mathbf{F}) \cdot \hat{n}\, dS = \frac{(\nabla \times \mathbf{F}) \cdot \nabla g}{|\nabla g \cdot \hat{k}|}\, dA = \frac{\langle z, 4z, 0 \rangle \cdot \langle 1, 0, 2 \rangle}{2}\, dxdy = \frac{1}{2}z$$

Since on the plane $z = 3 - \frac{x}{2}$, we can convert the integrand to be in terms of xy:

$$\frac{1}{2}z = \frac{1}{2}(3 - \frac{x}{2})$$

The region of integration on the xy-plane is given by the x and y coordinates of C. Since $x = 2\cos t$ and $y = \sin t$, this traces out a circle of radius 2 centered at the origin in the xy-plane.

Thus,

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int\int_{\mathcal{S}} (\nabla \times \mathbf{F}) \cdot \hat{n}\, dS = \iint_R (\frac{3}{2} - \frac{x}{4})\, dA$$

$$= \int_0^{2\pi} \int_0^2 \left(\frac{3r}{2} - \frac{1}{4} r^2 \cos\theta \right) \, dr \, d\theta$$

$$= \int_0^{2\pi} (3 - \frac{2}{3} \cos\theta) \, d\theta = 3(2\pi) - 0 = \boxed{6\pi}$$

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End Of Exam
