APPM 2350—Exam 3

Wednesday, Nov 17th 6:30pm-8pm 2021

This exam has 5 problems. Please start each new problem at the top of a new page in your blue book. Show all your work in your blue book and simplify your answers. Answers with no justification will receive no points. You are allowed one 8.5×11-in page of notes (ONE side). NO calculators, smartphones/watches, or any other electronic devices allowed.

Problem 1 (17 pts)
A wire lies along the intersection of the surfaces $z = \frac{1}{3}xy$ and $y = \frac{1}{2}x^2$ from $(0, 0, 0)$ to $(4, 8, \frac{32}{3})$. Suppose the charge density at any point on the wire is given by

$$\delta(x, y, z) = x \ \frac{Coulombs}{meter}$$

Find the **total charge** on the wire. (You can assume distance in xyz-space is measured in meters).

SOLUTION:

$$\text{Charge } = \int_{\mathcal{C}} \delta(x, y, z)\, ds = \int_{\mathcal{C}} x\, ds$$

where $\mathcal{C}$ is the path traced out by the wire.
To evaluate, we first need to parameterize the wire.
One simple parameterization:

$$\mathbf{r}(t) = \langle t, \frac{1}{2}t^2, \frac{1}{6}t^3\rangle, \quad 0 \le t \le 4$$

$$\text{Charge} = \int_0^4 t\, ||\mathbf{r}'(t)||\, dt$$

$$\mathbf{r}'(t) = \langle 1, t, \frac{1}{2}t^2\rangle$$

$$\implies ||\mathbf{r}'(t)|| = \sqrt{1 + t^2 + \frac{1}{4}t^4} = \sqrt{\left(1 + \frac{1}{2}t^2\right)^2} = |1 + \frac{1}{2}t^2| = 1 + \frac{1}{2}t^2$$

Thus

$$\text{Charge} = \int_0^4 t\left(1 + \frac{1}{2}t^2\right)\, dt = \int_0^4 \left(t + \frac{1}{2}t^3\right)\, dt = \left(\frac{t^2}{2} + \frac{t^4}{8}\right)\Bigg|_0^4 = 8 + 32 = 40 \text{ Coulombs}$$

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Problem 2 (17 points)
Consider the integral

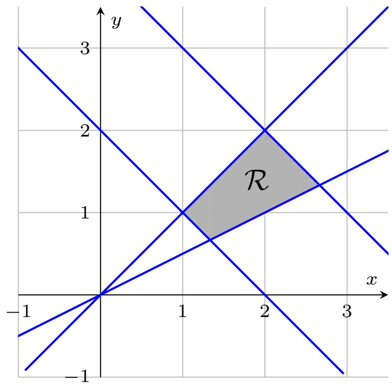
$$\iint_{\mathcal{R}} \left(\frac{x}{y} + 1\right)^2 e^{(x+y)(x/y)}\, dxdy$$

where $\mathcal{R}$ is the region in the xy-plane bounded by the curves

$$1 = \frac{x}{y} \qquad 2 = \frac{x}{y} \qquad y = 2 - x \qquad y = 4 - x$$

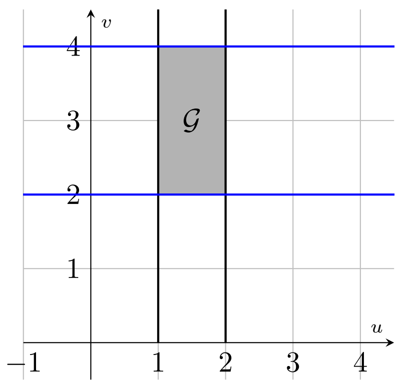
(a) Sketch the region of integration $\mathcal{R}$ in the xy-plane.
(b) Use an appropriate uv-transformation to rewrite this integral as **one** equivalent double integral in terms of u and v. (**Fully set-up but do not evaluate**). Be sure to include a sketch of the corresponding region of integration in the uv-plane as part of your solution.

SOLUTION:

(a) 

(b) We will use the following substitution:

$$u = \frac{x}{y} \qquad\qquad v = x + y$$



Solving for the xy-transformation yields:

$$x = \frac{uv}{1+u} \qquad\qquad y = \frac{v}{1+u}$$

Next compute the Jacobian:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{v}{(1+u)^2} & \frac{u}{1+u} \\ -\frac{v}{(1+u)^2} & \frac{1}{1+u} \end{vmatrix} = \frac{v}{(1+u)^2}$$

Making the substitution transforms the integral.

$$\int_2^4 \int_1^2 (u+1)^2 e^{uv} \frac{v}{(1+u)^2}\, dudv = \int_2^4 \int_1^2 ve^{uv}\, dudv$$

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Problem 3 (17 pts)
Suppose the temperature at any point in space is given by

$$T(x,y,z) = 3xyz$$

Let $\mathcal{R}$ be the triangle in the xy-plane with vertices $(0,0), (1,2), (-3,2)$ (including the interior). Set-up (**do not evaluate**) integral(s) to find the **average temperature** on the part of the surface

$$z = xy$$

that lies directly above the region $\mathcal{R}$. (By set-up we mean fully simply the integrand(s) and set-up the bounds of integration using the order of integration that leads to the fewest and simplest integrals).

<u>**SOLUTION:**</u>

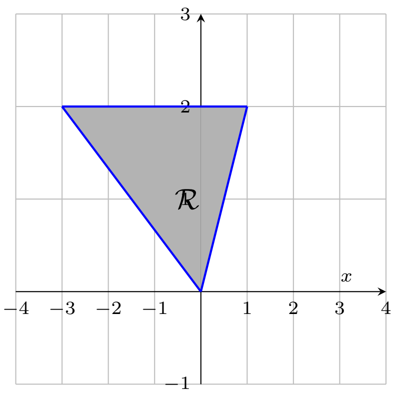
$$T_{ave} = \frac{\iint\limits_{\mathcal{S}} T(x,y,z)\, dS}{\iint\limits_{\mathcal{S}} dS}$$

Let $g(x, y, z) = xy - z$
Since the surface is given as a function of x and y it's easiest to project the surface onto the xy-plane Thus, let $\hat{p} = \mathbf{k}$.

$$dS = \frac{||\nabla g||}{|\nabla g \cdot \mathbf{k}|}\, dA = \frac{\sqrt{y^2 + x^2 + 1}}{|1|}\, dA$$

$$T_{ave} = \frac{\iint\limits_{\mathcal{S}} 3xyz\, dS}{\iint\limits_{\mathcal{S}} dS} = \frac{\iint\limits_{\mathcal{R}} 3xyz\sqrt{x^2 + y^2 + 1}\, dA}{\iint\limits_{\mathcal{R}} \sqrt{x^2 + y^2 + 1}\, dA} = \frac{\iint\limits_{\mathcal{R}} 3xy(xy)\sqrt{x^2 + y^2 + 1}\, dA}{\iint\limits_{\mathcal{R}} \sqrt{x^2 + y^2 + 1}\, dA}$$

where we have used the fact that $z = xy$ to rewrite the integrand in terms of x and y.
where $\mathcal{R}$ is the triangle in the xy-plane given by:



Thus $\mathcal{R}$ is bounded by $y = 2x$, $y = -\frac{2}{3}x$ and $y = 2$.
To set-up using the fewest number of integrals we use the ordering $dx\, dy$

$$T_{ave} = \frac{\int_0^2 \int_{-3y/2}^{y/2} 3x^2y^2\sqrt{x^2 + y^2 + 1}\, dx\, dy}{\int_0^2 \int_{-3y/2}^{y/2} \sqrt{x^2 + y^2 + 1}\, dx\, dy}$$

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Problem 4 (32 pts)
The following integral gives the **mass** of an object in grams (where distance in space is measured in meters):

$$\int_0^{\pi/4} \int_a^{a\sqrt{3}} \int_a^{\sqrt{4a^2 - r^2}} r^2\, dz\, dr\, d\theta$$

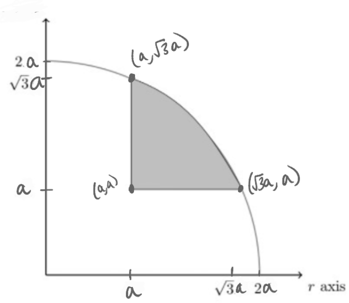
where a is a positive constant.

(a) What is the density, $\delta(x, y, z)$, of the object in Cartesian coordinates? Include units.
(b) Sketch and shade a 2D cross section of the object in rz-plane (for any θ such that $0 \le \theta \le \pi/4$). Label the (r, z) coordinates of all corners on the cross section.
(c) Sketch and shade the projection of the object onto the xy-plane. Label any intercepts.
(d) Set up, but DO NOT EVALUATE, equivalent integral(s) to find the **mass** of the object using
  (i) Cartesian coordinates in the order $dz\, dx\, dy$
  (ii) spherical coordinates in the order $d\rho\, d\phi\, d\theta$

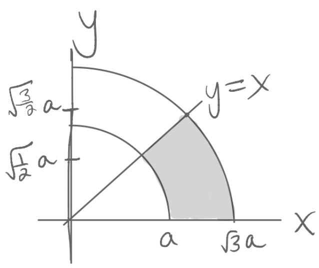
SOLUTION:

(a) Since the integrand is r^2, and we know the Jacobian for cylindrical is r, this means $\delta = r = \sqrt{x^2 + y^2}$

(b)



(c)



(d) (i)

$$Mass = \int_{0}^{\frac{a}{\sqrt{2}}} \int_{\sqrt{a^2-y^2}}^{\sqrt{3a^2-y^2}} \int_{a}^{\sqrt{4a^2-x^2-y^2}} \sqrt{x^2+y^2}\, dz\, dx\, dy + \int_{\frac{a}{\sqrt{2}}}^{a\sqrt{\frac{3}{2}}} \int_{y}^{\sqrt{3a^2-y^2}} \int_{a}^{\sqrt{4a^2-x^2-y^2}} \sqrt{x^2+y^2}\, dz\, dx\, dy$$

(ii)

$$Mass = \int_{0}^{\frac{\pi}{4}} \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \int_{a\csc\phi}^{2a} \rho^3 \sin^2\phi\, d\rho\, d\phi\, d\theta + \int_{0}^{\frac{\pi}{4}} \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \int_{a\sec\phi}^{2a} \rho^3 \sin^2\phi\, d\rho\, d\phi\, d\theta$$

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Problem 5 (17 points)
A neighborhood sits in the region in the xy-plane given by

$$x^2 + y^2 \le 4 \text{ and } x \ge 1$$

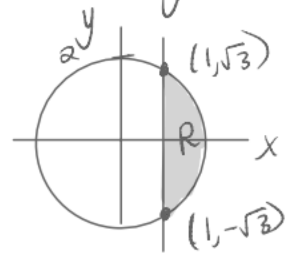
(where distances in the xy-plane are measured in miles). An earthquake occurs with epicenter at the origin. Suppose *at each point in the neighborhood*, the energy density released from the earthquake is given by the function

$$E(x,y) = \frac{10^6}{(d(x,y))^3} \qquad \frac{joules}{miles^2}$$

where $d(x,y)$ is the distance from (x,y) to the epicenter.
Find the **total amount of energy** (in joules) released by the earthquake in this neighborhood.

SOLUTION:

Total energy $= \iint_R E(x,y)\,dA$ where R is region



$d = \sqrt{x^2+y^2}$

thus $E(x,y) = \dfrac{10^6}{\left(\sqrt{x^2+y^2}\right)^3}$

Switch to polar:

$$\text{Total energy} = 10^6 \int_{\theta=-\frac{\pi}{3}}^{\theta=\frac{\pi}{3}} \int_{r=\sec\theta}^{r=2} \frac{1}{r^3}\, r\, dr\, d\theta$$

$$= 10^6 \int_{-\pi/3}^{\pi/3} \int_{\sec\theta}^{2} r^{-2}\, dr\, d\theta = 10^6 \int_{-\pi/3}^{\pi/3} \left[-\frac{1}{r} \Big|_{r=\sec\theta}^{r=2} \right] d\theta$$

$$= 10^6 \int_{-\pi/3}^{\pi/3} \left[\cos\theta - \frac{1}{2}\right] d\theta = 10^6 \left[\sin\theta - \frac{1}{2}\theta \Big|_{-\pi/3}^{\pi/3}\right]$$

$$= 10^6 \left[\underbrace{\sin(\pi/3)}_{\sqrt{3}/2} - \frac{\pi}{6} - \underbrace{\sin(-\pi/3)}_{-\frac{\sqrt{3}}{2}} - \frac{\pi}{6}\right] = \boxed{10^6\left[\sqrt{3} - \frac{\pi}{3}\right] \text{ Joules}}$$

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