APPM 2350—Exam 2

Wednesday Oct 20th, 6:30pm-8pm 2021

This exam has 4 problems. Please start each new problem at the top of a new page in your blue book. Show all your work in your blue book and simplify your answers. Answers with no justification will receive no points. You are allowed one 8.5×11-in page of notes (ONE side). NO calculators, smartphones/watches, or any other electronic device.

Problem 1 (20 pts)
The following parts are not related

(a) Given

$$g(x, y, z) = x^2 + y^2 + z + z^3$$

find an equation of the tangent plane to the surface $g(x, y, z) = 7$ at the point $(-1, 4, -2)$. Give your answer in standard (linear) form.

SOLUTION:

Since the gradient of a function is normal to its level sets, $\nabla g(x, y, z)$ will be normal to the level surface $g(x, y, z) = 7$.
Thus the tangent plane to the surface $g(x, y, z) = 7$ at the point $(-1, 4, -2)$ will be the plane with normal vector $\nabla g(-1, 4, -2)$ that contains that point $(-1, 4, -2)$.

$$\nabla g(-1, 4, -2) = \left\langle 2x, 2y, 1 + 3z^2 \right\rangle \Big|_{(-1,4,-2)} = \langle -2, 8, 13 \rangle = \vec{n}$$

Thus the tangent plane is given by

$$-2(x + 1) + 8(y - 4) + 13(z + 2) = 0$$

$$\implies \boxed{-2x + 8y + 13z = 8}$$

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(b) Prove the following limit does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{2x^2 y}{x^4 + y^2}$$

SOLUTION:

To prove the limit does not exist, we need to find 2 different paths that approach the origin, where the limit at the origin is different.
Path 1:
Choose any line through the origin, i.e. let $y = mx$, $m \neq 0$ and take the limit as $x \to 0$:

$$\lim_{x\to 0} \frac{2x^2 mx}{x^4 + (mx)^2} = \lim_{x\to 0} \frac{2x^2 mx}{x^2(x^2 + m^2)} = \lim_{x\to 0} \frac{2mx}{x^2 + m^2} = \frac{0}{m^2} = \boxed{0}$$

Path 2:
Choose a parabola through the origin, i.e. let $y = x^2$ and take the limit as $x \to 0$:

$$\lim_{x\to 0} \frac{2x^2(x^2)}{x^4 + (x^2)^2} = \lim_{x\to 0} \frac{2x^4}{2x^4} = \lim_{x\to 0} 1 = \boxed{1}$$

$\boxed{\text{Since there are different limits on different curves approaching the origin, the limit at the origin does not exist}}$.

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Problem 2 (30 points)
Given

$$U(x,y) = 1 + xy, \text{ where } x > 0 \text{ and } y > 0$$

.

(a) Give the equation of the level curve of $U(x,y)$ that passes through the point $(1,2)$.

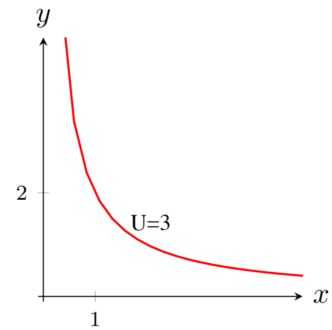
(b) Sketch the level curve you found in part (a). Label the value of U on this curve and label any intercepts.

(c) Give a parameterization of a path, $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ (where $x(t) > 0$ and $y(t) > 0$), such that $\frac{dU}{dt} = 0$ for all t.

(d) Suppose you have 90 dollars to spend on lecture notes, costing 3 dollars a page, and coffee, costing 5 dollars a cup. You decide to buy x pages of lecture notes and y cups of coffee. Suppose your total satisfaction from the purchases is given by $U(x,y)$ (called the *utility function* by economists). Use Lagrange multipliers to determine how many lecture notes and cups of coffee you should purchase to maximize your satisfaction.

<u>**SOLUTION:**</u>

(a) Level curves are given by $U(x,y) = xy + 1 = c$. The level curve passing through $(1,2)$ has a value $c = U(1,2) = 3$. Thus this curve is given by $3 = xy + 1 \implies y = \frac{2}{x}$.



(b)

(c) Since $U(x,y) = c$ along any level curve, $\frac{d}{dt}\Big(U(x(t),y(t))\Big) = 0$ along any level curve. For example, we can just parameterize the level curve from part (a).

$\boxed{\text{One possible parameterization: } \mathbf{r}(t) = \left\langle t, \frac{2}{t} \right\rangle,\ t > 0}$

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(d) We need to find the maximum and minimum values of $U(x,y)$ subject to the constraint $g(x,y) = 3x + 5y = 90$. With $U_x = y, U_y = x, g_x = 3, g_y = 5$, we have the following system of equations:

$$y = 3\lambda$$
$$x = 5\lambda$$
$$3x + 5y = 90$$

Solving for λ in the first two equations yields $\frac{y}{3} = \frac{x}{5} \implies y = \frac{3}{5}x$.

Plugging this into the constraint we get $3x + 5(\frac{3x}{5}) = 90 \implies x = 15$. Thus $y = \frac{3}{5}x = 9$.

This is the only critical point found by Lagrange multipliers. To prove it is a max, we can choose any other point that satisfies the constraint (for example $(x,y) = (30,0)$ Notice that $U(30,0) = 31 < U(15,9) = 136$, so the point we found is in fact a max.

$\boxed{\text{Thus to maximize your satisfaction you should buy 15 pages of lecture notes and 9 cups of coffee}}$.

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Problem 3 (35 points)
Suppose the temperature at the point (x,y) on the xy-plane is given by

$$T(x,y) = 3x^2 + 3x^2y + 6y^2$$

where temperature is in Celsius and distance is in feet.

(a) Find and classify all critical points of $T(x, y)$.
(b) If you start at the location $(x, y) = (1, 0)$ and you move along a straight path toward the point $(x, y) = (4, 4)$, use a directional derivative to estimate how far you'd need to move in this direction until the temperature reaches $5°$C. Include units.
(c) At the point $(x, y) = (1, 0)$, is there a direction, $\mathbf{u}$, in which the rate of change of the temperature function, $T(x, y)$, equals $8 \frac{°C}{ft}$? If so, find $\mathbf{u}$. If not, explain why not.
(d) Suppose an ant is moving in the xy-plane along a path parameterized by $\mathbf{r}(t)$ where t is in minutes. You are given the following information about the ant's path:

$$\mathbf{r}(1) = 3\mathbf{i} + 4\mathbf{j} \qquad \mathbf{v}(1) = 1\mathbf{i} - \frac{1}{2}\mathbf{j}$$

$$\mathbf{r}(3) = 1\mathbf{i} \qquad \mathbf{v}(3) = 2\mathbf{i} - 5\mathbf{j}$$

Find the instantaneous rate of change of the temperature T with respect to time along the ant's path at the point $(x, y) = (1, 0)$. Include units.

SOLUTION:

(a) The critical points are located where both first-order partial derivatives equal zero. In other words, each critical point satisfies both of the following equations:

$$T_x(x, y) = 6x + 6xy = 0 \implies 6x(1 + y) = 0$$
$$T_y(x, y) = 3x^2 + 12y = 0$$

The first equation can only be satisfied by $x = 0$ or $y = -1$.

Plugging $x = 0$ into the second equation implies that $y = 0$. Plugging $y = -1$ into the second equation implies $x = \pm 2$

Therefore, the three critical points are:

$$\boxed{(0, 0), (2, -1), (-2, -1)}$$

To classify the critical points, apply the Second Derivative Test. The metric used in the test is:

$$\begin{aligned} D(x, y) &= [T_{xx}(x, y)][T_{yy}(x, y)] - [T_{xy}(x, y)]^2 \\ &= (6 + 6y)(12) - (6x)^2 \end{aligned}$$

For the critical point $(0, 0)$, we have:

$$D(0, 0) = (6)(12) > 0 \quad \Rightarrow \quad \text{Relative extremum at } (0, 0)$$
$$T_{xx}(0, 0) = 6 > 0 \quad \Rightarrow \quad \boxed{\text{Relative minimum at } (0, 0)}$$

For the critical point $(2, -1)$, we have:

$$D(2, -1) = (0)(12) - 12^2 < 0 \quad \Rightarrow \quad \boxed{\text{Saddle point at } (2, -1)}$$

For the critical point $(-2, -1)$, we have:

$$D(-2, -1) = (0)(12) - 12^2 < 0 \quad \Rightarrow \quad \boxed{\text{Saddle point at } (-2, -1)}$$

(b) We start by finding the directional derivative at the point $(1, 0)$ in the direction $\mathbf{u} = \frac{\langle 4-1, 4-0 \rangle}{\sqrt{3^2+4^2}} = \frac{\langle 3,4 \rangle}{5}$

$$\nabla T(1, 0) = \langle 6, 3 \rangle$$

$$\frac{dT}{ds} = D_uT(1,0) = \nabla T(1,0)\cdot \mathbf{u} = \langle 6,3\rangle \cdot \frac{\langle 3,4\rangle}{5} = 6 \ \ \frac{{}^\circ\mathrm{C}}{ft}$$

Starting from the point $(1,0)$ we want to determine how far we need to travel in the direction $\mathbf{u}$ for the temperature to reach 5°

Notice $T(1,0) = 3$.

Thus, want to solve

$$\begin{aligned} 3 + \frac{dT}{ds}\Delta s &= 5 \\ \implies 6\Delta s &= 2 \\ \implies \boxed{\Delta s = \frac{1}{3}\ ft} \end{aligned}$$

(c)

$$D_uT(1,0) = \nabla T(1,0)\cdot \mathbf{u} = ||\nabla T(1,0)|| \cos\theta \le ||\nabla T(1,0)|| = \sqrt{6^2+3^2} = \sqrt{45}$$

.

$\boxed{\text{Thus since } D_uT(1,0) \le \sqrt{45} < 8 \text{ there is no direction in which } D_uT = 8}$

(d) Using the chain rule,

$$\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt}$$

$$\left\langle \frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}\right\rangle\Bigg|_{(x,y)=(1,0)} = \nabla T(1,0) = \langle 6,3\rangle$$

$$\mathbf{r}(3) = \mathbf{i} \implies t = 3 \text{ when } (x,y) = (1,0)$$

$$\left\langle \frac{dx}{dt}, \frac{dy}{dt}\right\rangle\Bigg|_{t=3} = \mathbf{v}(3) = \langle 2,-5\rangle$$

Thus

$$\begin{aligned} \frac{dT}{dt} &= \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} \\ &= (6)(2) + (3)(-5) = \boxed{-3\ \ \frac{{}^\circ\mathrm{C}}{min}} \end{aligned}$$

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Problem 4 (15 points)

Let $G(x,y)$ be a continuous function with continuous partial derivatives such that

$$G(1,0) = -23,\quad \frac{\partial G}{\partial x}(1,0) = -2,\quad \frac{\partial G}{\partial y}(1,0) = -5,\quad \frac{\partial^2 G}{\partial x^2}(1,0) = 4,\quad \frac{\partial^2 G}{\partial y^2}(1,0) = 8,$$
$$\frac{\partial^2 G}{\partial x \partial y}(1,0) = \frac{\partial^2 G}{\partial y \partial x}(1,0) = -3,$$

(a) Given this information, find a 2nd order (i.e. quadratic) Taylor approximation of $G(x,y)$ and use it to approximate the value of $G(3,-1)$.

(b) Suppose $\left|\frac{\partial^3 G}{\partial x^3}\right| < \frac{1}{4}$, $\quad \left|\frac{\partial^3 G}{\partial y^3}\right| < \frac{1}{4}$, $\quad \left|\frac{\partial^3 G}{\partial x \partial y^2}\right| < \frac{1}{4}$, and $\left|\frac{\partial^3 G}{\partial y \partial x^2}\right| < \frac{1}{4}$ for all (x,y).

Let $|y| \le 1.5$. Use Taylor's error bound to find the largest interval of x values for which the absolute value of the error of the 2nd order Taylor approximation of $G(x,y)$ centered at $(1,0)$ is less than or equal to $\frac{1}{3}$.

<u>**SOLUTION:**</u>

(a) The quadratic Taylor approximation of $G(x,y)$ centered at the point $(1,0)$ is given by:

$$Q(x,y) = G(1,0) + G_x(1,0)(x-1) + G_y(1,0)y + \frac{1}{2}(G_{xx}(1,0)(x-1)^2 + 2G_{xy}(1,0)(x-1)y + G_{yy}(1,0)y^2)$$

$$\boxed{= -23 - 2(x-1) - 5y + 2(x-1)^2 + 4y^2 - 3(x-1)y}$$

Thus

$$\begin{aligned} G(3,-1) &\approx Q(3,-1) \\ &= -23 - 2(2) - 5(-1) + 2(2)^2 + 4(-1)^2 - 3(2)(-1) \\ &\boxed{= -4} \end{aligned}$$

(b) Taylor's error formula for a quadratic approximation is given by:

$$|E(x,y)| \le \frac{M}{3!}\left(\left|x-1\right| + \left|y\right|\right)^3$$

where M is an upper bound on the function's 3rd order mixed partial derivatives on the region.

We are given

$$M = \frac{1}{4}, \qquad |y| \le 1.5 \qquad \text{and } |E(x,y)| \le \frac{1}{3}$$

Thus

$$\begin{aligned} &\frac{1}{24}\left(\left|x-1\right| + 1.5\right)^3 \le \frac{1}{3} \\ \implies &\left(\left|x-1\right| + 1.5\right)^3 \le 8 \\ \implies &\left(\left|x-1\right| + 1.5\right) \le 2 \\ \implies &\boxed{\left|x-1\right| \le 0.5} \end{aligned}$$

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End Of Exam