APPM 2350—Exam 2

Wednesday Oct 20th, 6:30pm-8pm 2021

This exam has 4 problems. Please start each new problem at the top of a new page in your blue book. Show all your work in your blue book and simplify your answers. Answers with no justification will receive no points. You are allowed one 8.5×11-in page of notes (ONE side). NO calculators, smartphones/watches, or any other electronic device.

Problem 1 (20 pts)
The following parts are not related

(a) Given

$$g(x,y,z) = x^2 + y^2 + z + z^3$$

find an equation of the tangent plane to the surface $g(x,y,z) = 7$ at the point $(-1,4,-2)$. Give your answer in standard (linear) form.

(b) Prove the following limit does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2}$$

Problem 2 (30 points)
Given

$$U(x,y) = 1 + xy, \text{ where } x > 0 \text{ and } y > 0$$

.

(a) Give the equation of the level curve of $U(x,y)$ that passes through the point $(1,2)$.

(b) Sketch the level curve you found in part (a). Label the value of U on this curve and label any intercepts.

(c) Give a parameterization of a path, $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ (where $x(t) > 0$ and $y(t) > 0$), such that $\frac{dU}{dt} = 0$ for all t.

(d) Suppose you have 90 dollars to spend on lecture notes, costing 3 dollars a page, and coffee, costing 5 dollars a cup. You decide to buy x pages of lecture notes and y cups of coffee. Suppose your total satisfaction from the purchases is given by $U(x,y)$ (called the *utility function* by economists). Use Lagrange multipliers to determine how many lecture notes and cups of coffee you should purchase to maximize your satisfaction.

Problem 3 (35 points)
Suppose the temperature at the point (x,y) on the xy-plane is given by

$$T(x,y) = 3x^2 + 3x^2y + 6y^2$$

where temperature is in Celsius and distance is in feet.

(a) Find and classify all critical points of $T(x,y)$.

(b) If you start at the location $(x,y) = (1,0)$ and you move along a straight path toward the point $(x,y) = (4,4)$, use a directional derivative to estimate how far you'd need to move in this direction until the temperature reaches $5°$C. Include units.

(c) At the point $(x,y) = (1,0)$, is there a direction, $\mathbf{u}$, in which the rate of change of the temperature function, $T(x,y)$, equals $8\frac{°C}{ft}$? If so, find $\mathbf{u}$. If not, explain why not.

(d) Suppose an ant is moving in the xy-plane along a path parameterized by $\mathbf{r}(t)$ where t is in minutes. You are given the following information about the ant's path:

$$\mathbf{r}(1) = 3\mathbf{i} + 4\mathbf{j} \quad \mathbf{v}(1) = 1\mathbf{i} - \frac{1}{2}\mathbf{j}$$

$$\mathbf{r}(3) = 1\mathbf{i} \quad \mathbf{v}(3) = 2\mathbf{i} - 5\mathbf{j}$$

Find the instantaneous rate of change of the temperature T with respect to time along the ant's path at the point $(x,y) = (1,0)$. Include units.

CONT'D ON NEXT PAGE

Problem 4 (15 points)

Let $G(x,y)$ be a continuous function with continuous partial derivatives such that

$$G(1,0)=-23,\quad \frac{\partial G}{\partial x}(1,0)=-2,\quad \frac{\partial G}{\partial y}(1,0)=-5,\quad \frac{\partial^2 G}{\partial x^2}(1,0)=4,\quad \frac{\partial^2 G}{\partial y^2}(1,0)=8,$$
$$\frac{\partial^2 G}{\partial x \partial y}(1,0)=\frac{\partial^2 G}{\partial y \partial x}(1,0)=-3,$$

(a) Given this information, find a 2nd order (i.e. quadratic) Taylor approximation of $G(x,y)$ and use it to approximate the value of $G(3,-1)$.

(b) Suppose $\left|\frac{\partial^3 G}{\partial x^3}\right| < \frac{1}{4}$, $\left|\frac{\partial^3 G}{\partial y^3}\right| < \frac{1}{4}$, $\left|\frac{\partial^3 G}{\partial x \partial y^2}\right| < \frac{1}{4}$, and $\left|\frac{\partial^3 G}{\partial y \partial x^2}\right| < \frac{1}{4}$ for all (x,y).

Let $|y| \le 1.5$. Use Taylor's error bound to find the largest interval of x values for which the absolute value of the error of the 2nd order Taylor approximation of $G(x,y)$ centered at $(1,0)$ is less than or equal to $\frac{1}{3}$.