APPM 2350—Exam 1

Wednesday Sep 22nd, 6:30pm-8pm 2021

Please start each new problem at the top of a new page in your blue book. Show all your work in your blue book and simplify your answers. Answers with no justification will receive no points. You are allowed one 8.5×11-in page of notes (ONE side). You may NOT use a calculator, smartphone, smartwatch, the Internet or any other electronic device.

Problem 1 (18 points)
Consider the three points $P(0,3,1), Q(1,2,-1), R(-1,-2,0)$.

(a) Find the equation of the plane containing the points. Write your answer in standard (i.e. linear) form.
(b) Find the (acute) angle the plane makes with the yz-plane.

SOLUTION:

(a) We need a point on the plane (we have three actually) and a normal vector $\mathbf{n}$ to the plane. Using $\overrightarrow{PQ} = \langle 1,-1,-2\rangle$ and $\overrightarrow{PR} = \langle -1,-5,-1\rangle$ we can get a normal vector as follows:

$$\begin{aligned}\mathbf{n} &= \overrightarrow{PQ} \times \overrightarrow{PR} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -2 \\ -1 & -5 & -1 \end{vmatrix} \\ &= \cdots = \langle -9, 3, -6\rangle \qquad \text{or} \qquad \langle 3,-1,2\rangle\end{aligned}$$

since any scalar multiple will do. (You don't have to do this, the answer will work out the same, but perhaps it's easier this way)

Using the point P, the equation of the plane is therefore

$$\begin{aligned} 3(x-0) - 1(y-3) + 2(z-1) &= 0 \\ 3x - y + 3 + 2z - 2 &= 0 \\ 3x - y + 2z &= -1 \end{aligned}$$

(b) A normal vector to the yz-plane is $\mathbf{i} = \langle 1,0,0\rangle$ actually, so the angle between the plane in part (a) and the yz-plane is thus

$$\begin{aligned}\arccos\left(\frac{|\mathbf{n}\cdot\mathbf{i}|}{\|\mathbf{n}\|\|\mathbf{i}\|}\right) &= \arccos\left(\frac{|3|}{\sqrt{3^2+(-1)^2+2^2}(1)}\right) \\ &= \arccos\left(\frac{3}{\sqrt{14}}\right)\end{aligned}$$

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Problem 2 (20 points)
Consider the surface $4x - 4y^2 + z^2 = 0$.

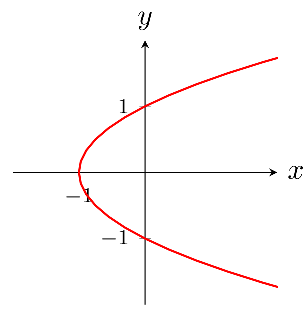
(a) Name this surface.
(b) Find the equation of the trace of surface in the plane $z = 2$
(c) Sketch the trace you found in part(b) in 2D. Label any intercepts.
(d) Find the coordinates of the point(s) where the line with the following symmetric equations intersects the surface.

$$\frac{x}{3} = 3 - y = \frac{z-1}{2}$$

SOLUTION:

(a) Rearranging we can write this in a perhaps more revealing way as $x = y^2 - \frac{1}{4}z^2$, showing that this is a hyperbolic paraboloid.
(b) Setting $z = 2$ in the equation gives $4x - 4y^2 + 4 = 0 \implies x = y^2 - 1$

(c) Sketch.



(d) The parametric equations of the line are

$$x(t) = 3t$$
$$y(t) = 3 - t$$
$$z(t) = 1 + 2t$$

For the line to intersect the surface, its equation must satisfy that of the surface:

$$\begin{gathered} 4(3t) - 4(3-t)^2 + (1+2t)^2 = 0 \\ 12t - 4(9 - 6t + t^2) + 1 + 4t + 4t^2 = 0 \\ 12t - 36 + 24t - 4t^2 + 1 + 4t + 4t^2 = 0 \\ 40t - 35 = 0 \\ t = \frac{7}{8} \end{gathered}$$

The line intersects the surface at $\left(3 \cdot \frac{7}{8}, 3 - \frac{7}{8}, 1 + 2 \cdot \frac{7}{8}\right) = \left(\frac{21}{8}, \frac{17}{8}, \frac{11}{4}\right)$

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Problem 3 (20 points)

A jet travels along the curve of intersection of the surfaces $z = \frac{1}{3}xy$ and $y = \frac{1}{2}x^2$. Assume distances are measured in miles.

(a) Find the total distance the jet travels along this curve starting from the origin and ending at the point $(6, 18, 36)$

(b) Suppose at the point $(2, 2, \frac{4}{3})$ along its path, the jet fires a missile straight ahead at a speed of 12 miles per second. What is the location of the missile 5 seconds after it is fired?

SOLUTION:

(a) To find the arclength, we first need to parameterize the curve.
One simple parameterization:

$$\mathbf{r}(t) = \langle t, \frac{1}{2}t^2, \frac{1}{6}t^3 \rangle, \qquad 0 \le t \le 6$$

$$arclength = \int_0^6 ||\mathbf{r}'(t)||\, dt$$

$$\mathbf{r}'(t) = \langle 1, t, \frac{1}{2}t^2 \rangle$$

$$\implies ||\mathbf{r}'(t)|| = \sqrt{1 + t^2 + \frac{1}{4}t^4} = \sqrt{\left(1 + \frac{1}{2}t^2\right)^2} = |1 + \frac{1}{2}t^2| = 1 + \frac{1}{2}t^2$$

Thus

$$arclength = \int_0^6 \left(1 + \frac{1}{2}t^2\right)\, dt = \left(t + \frac{t^3}{6}\right)\Bigg|_0^6 = 6 + \frac{6^3}{6} = 42 \text{ miles}$$

(b) The missile will travel along the tangent line to the jet at the point $(2, 2, \frac{4}{3})$.

To find the equation of this tangent line, we need a point on the line and and a vector in the direction of the missile.

Point on the tangent line: $(2, 2, \frac{4}{3})$

Vector in the direction of the missile= $\mathbf{r}'(2) = \langle 1, 2, 2 \rangle$ (using our parameterization from part (a)).

However, notice that the speed of $\mathbf{r}'(2)$ is not the same as the missile's speed.

The missle's velocity vector = $(12)\frac{\mathbf{r}'(2)}{||\mathbf{r}'(2)||} = 12\frac{\langle 1,2,2 \rangle}{3} = \langle 4, 8, 8 \rangle$

Thus, the missile will travel along the line given below for $t \geq 0$:

$$x = 2 + 4t$$

$$y = 2 + 8t$$

$$z = \frac{4}{3} + 8t$$

After traveling along this line for 5 seconds, the missile will be at the point:

$$x = 2 + 4(5) = 22$$

$$y = 2 + 8(5) = 42$$

$$z = \frac{4}{3} + 8(5) = \frac{124}{3}$$

$$\Longrightarrow (x, y, z) = \left(22, 42, \frac{124}{3}\right)$$

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Problem 4 (20 points)

A particle travels along a curve parameterized by

$$\mathbf{r}(t) = \langle 4t, \cos(3t), \sin(3t) \rangle, \qquad 0 \leq t \leq \pi/3$$

where t is time.

(a) At what coordinates (x, y, z), if any, is the particle's unit normal vector, $\mathbf{N}(t)$ parallel to the following plane? Explain/justify your answer.

$$\frac{3}{2}x + y - \sqrt{3}z = \frac{2\pi}{3}$$

(b) At what time(s), if any, is the curvature of the particle's path equal to $\frac{1}{2}$? Explain/justify your answer.

<u>**SOLUTION:**</u>

(a)

$$\mathbf{r}'(t) = \langle 4, -3\sin(3t), 3\cos(3t) \rangle$$

$$||\mathbf{r}'(t)|| = \sqrt{16 + 9\sin^2(3t) + 9\cos^2(3t)} = 5$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||} = \frac{1}{5}\langle 4, -3\sin(3t), 3\cos(3t) \rangle$$

$$\frac{d}{dt}\mathbf{T}(t) = \frac{1}{5}\langle 0, -9\cos(3t), -9\sin(3t) \rangle$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{||\mathbf{T}'(t)||} = \langle 0, -\cos(3t), -\sin(3t) \rangle$$

The normal vector to the plane is $\mathbf{n} = \langle \frac{3}{2}, 1, -\sqrt{3} \rangle$. $\mathbf{N}(t)$ parallel to the plane means that $\mathbf{N}(t)$ is orthogonal to $\mathbf{n}$. $\mathbf{N} \cdot \mathbf{n} = 0$ gives the following:

$$\begin{aligned}
-\cos(3t) + \sqrt{3}\sin(3t) &= 0 \\
\tan(3t) &= \frac{1}{\sqrt{3}} \\
3t &= \frac{\pi}{6} + 2\pi k \\
t &= \frac{\pi}{18} + \frac{2}{3}\pi k \\
t &= \frac{\pi}{18} \quad \text{(the only solution within the given interval.)}
\end{aligned}$$

The position of the particle at $t = \frac{\pi}{18}$ is then

$$\mathbf{r}\left(\frac{\pi}{18}\right) = \left\langle \frac{2\pi}{9}, \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle.$$

Thus

$$(x, y, z) = \left(\frac{2\pi}{9}, \frac{\sqrt{3}}{2}, \frac{1}{2} \right)$$

(b)

$$\begin{aligned}
\kappa(t) &= \frac{||\mathbf{T}'(t)||}{||\mathbf{r}'(t)||} \\
&= \frac{\frac{9}{5}}{5} = \frac{9}{25}
\end{aligned}$$

The curvature is constant and always equal to $\frac{9}{25}$. There are no points where curvature is equal to $\frac{1}{2}$.

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Problem 5 (22 pts)
A car starts at the point $(x, y) = (8, 1)$ when $t = 0$ and moves with **velocity** given by

$$\mathbf{v}(t) = c\langle 1, 2t - 2 \rangle$$

where $t \geq 0$ is the time and $c > 0$ is a constant to be determined.

(a) Find the car's acceleration vector, $\mathbf{a}(t)$.
(b) Find the car's position vector, $\mathbf{r}(t)$
(c) Calculate the normal scalar component of the acceleration, a_N, as a function of t.
(d) Find the time(s) when a_N is largest. Show work justifying your answer.
(e) As an actual industry standard, to avoid rolling over, the normal component of the acceleration for passenger cars must not exceed $1g$, where g is the gravitational acceleration. Find the maximum value of c such that the car won't rollover, as a function of g.

SOLUTION:

(a)

$$\mathbf{a}(t) = \langle 0, 2c \rangle$$

(b)

$$\mathbf{r}(t) = \langle ct + 8, ct^2 - 2ct + 1 \rangle$$

(c)

$$a_N = \frac{|\mathbf{a} \times \mathbf{v}|}{|\mathbf{v}|} = c\frac{|\langle 0, 2 \rangle \times \langle 1, 2t - 2 \rangle|}{|\langle 1, 2t - 2 \rangle|} = \frac{2c}{\sqrt{1 + 4(t - 1)^2}}$$

(d)

$$\begin{aligned} a_N' &= 0 \\ \left(\frac{2c}{\sqrt{1+4(t-1)^2}}\right)' &= 0 \\ \frac{8c(t-1)}{(1+4(t-1))^{3/2}} &= 0 \\ t &= 1 \quad \text{(5 pts)} \end{aligned}$$

$a_N' > 0$ when $t \to 1^-$ and $a_N' < 0$ when $t \to 1^+ \Rightarrow a_N(t=1) = 2c$ is a maximum. (2 pts)

(e) $a_N(t=1) = 2c = g$, therefore for the car not to roll over we need $c \leq \frac{g}{2}$

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