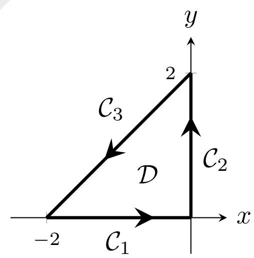
1. [APPM 2350 Exam (30 pts)] Consider the vector field $\mathbf{F} = -\frac{1}{2}y\,\mathbf{i} + \frac{1}{8}x^2y\,\mathbf{j}$. Let $\mathcal{C}$ be the triangle with vertices $(-2,0)$, $(0,0)$, $(0,2)$ oriented counterclockwise.

   (a) (15 pts) By direct calculation, find the circulation of $\mathbf{F}$ along $\mathcal{C}$. Hint: Visualizing the vector field will save some computational effort.

   (b) (15 pts) Using Green's Theorem, calculate the outward flux of $\mathbf{F}$ through $\mathcal{C}$.

SOLUTION:

Consider the following sketch.



(a) $\mathcal{C}$ consists of 3 piecewise smooth line segments. The circulation is given by

$$\oint_{\mathcal{C}} \mathrm{F}\cdot \mathrm{d}\mathbf{r} = \int_{\mathcal{C}_1} \mathrm{F}\cdot \mathrm{d}\mathbf{r} + \int_{\mathcal{C}_2} \mathrm{F}\cdot \mathrm{d}\mathbf{r} + \int_{\mathcal{C}_3} \mathrm{F}\cdot \mathrm{d}\mathbf{r}$$

To save computational effort, notice that on $\mathcal{C}_1$, $\mathbf{F} = \mathbf{0}$ so that there is no component of $\mathbf{F}$ along $\mathcal{C}_1$. Thus, $\int_{\mathcal{C}_1} \mathrm{F}\cdot \mathrm{d}\mathbf{r} = 0$. On $\mathcal{C}_2$, $\mathbf{F} = -\frac{1}{2}y\,\mathbf{i}$ so that there is no component of $\mathbf{F}$ along $\mathcal{C}_2$. Therefore, $\int_{\mathcal{C}_2} \mathrm{F}\cdot \mathrm{d}\mathbf{r} = 0$. On $\mathcal{C}_3$,

$$\mathbf{r}_3(t) = (1-t)\langle 0,2\rangle + t\langle -2,0\rangle = \langle -2t, 2-2t\rangle \quad 0 \le t \le 1$$

$$\mathbf{F}(\mathbf{r}_3(t)) = \left\langle -\frac{1}{2}(2-2t), \frac{1}{8}(-2t)^2(2-2t)\right\rangle = \langle t-1, t^2-t^3\rangle$$

$$\mathbf{r}_3'(t) = \langle -2,-2\rangle$$

$$\mathbf{F}(\mathbf{r}_3(t))\cdot \mathbf{r}_3'(t) = (-2)(t-1) - 2(t^2-t^3) = 2(1-t-t^2+t^3)$$

$$\int_{\mathcal{C}_3} \mathrm{F}\cdot \mathrm{d}\mathbf{r} = 2\int_0^1 \left(1-t-t^2+t^3\right)\mathrm{d}t = 2\left(t - \frac{t^2}{2} - \frac{t^3}{3} + \frac{t^4}{4}\right)\Bigg|_0^1 = 2\left(1-\frac{1}{2}-\frac{1}{3}+\frac{1}{4}\right) = \frac{5}{6}$$

and the circulation of $\mathbf{F}$ on $\mathcal{C}$ is $5/6$.

(b)

$$\text{Flux} = \oint_{\mathcal{C}} \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}s = \iint_{\mathcal{D}} \nabla\cdot\mathbf{F}\,\mathrm{d}A = \int_{-2}^{0}\int_0^{x+2} \frac{1}{8}x^2\,\mathrm{d}y\,\mathrm{d}x$$

$$= \frac{1}{8}\int_{-2}^{0} x^2 y\Big|_0^{x+2}\mathrm{d}x = \frac{1}{8}\int_{-2}^{0}\left(x^3+2x^2\right)\mathrm{d}x$$

$$= \frac{1}{8}\left(\frac{x^4}{4} + \frac{2}{3}x^3\right)\Bigg|_{-2}^{0} = \frac{1}{6}$$

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2. [APPM 2350 Exam (30 pts)] Consider the force field $\mathbf{F} = 4xe^z\,\mathbf{i} + \cos y\,\mathbf{j} + 2x^2e^z\,\mathbf{k}$.

   (a) (10 pts) Set up, but **do not evaluate**, the integral (in terms of t) to find the work done by the force field in moving an object along the curve $\mathcal{C} = \left(\sqrt{t}, \frac{\pi t}{2}, t^2\right)$ for $1 \le t \le 3$.

   (b) (15 pts) The integral in part (a) is rather messy but you still need to find the work. Use one of the important Calculus 3 theorems to actually compute the work.

(c) (5 pts) Now consider the curve $\mathcal{C}$ given by the intersection of the surfaces $x^2+y^2=1$ and $x+y-z=0$. Find the work done by the force field in moving an object along this curve. Hint: This can be done with **very** little computational effort.

SOLUTION:

(a)

$$\mathbf{r}(t)=\left\langle t^{1/2},\frac{\pi}{2}t,t^2\right\rangle \implies \mathbf{r}'(t)=\left\langle \frac{1}{2}t^{-1/2},\frac{\pi}{2},2t\right\rangle$$

$$\mathbf{F}(\mathbf{r}(t))=\left\langle 4\sqrt{t}e^{t^2},\cos\left(\frac{\pi}{2}t\right),2te^{t^2}\right\rangle$$

$$\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)=2e^{t^2}+\frac{\pi}{2}\cos\left(\frac{\pi}{2}t\right)+4t^2e^{t^2}$$

$$\text{Work}=\int_{\mathcal{C}}\mathbf{F}\cdot d\mathbf{r}=\int_1^3\left[2e^{t^2}+\frac{\pi}{2}\cos\left(\frac{\pi}{2}t\right)+4t^2e^{t^2}\right]dt$$

(b) If the force field is conservative, then we can use the Fundamental Theorem of Line Integrals. To check this

$$\nabla\times\mathbf{F}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ 4xe^z & \cos y & 2x^2e^z\end{vmatrix}=0\,\mathbf{i}+(4xe^z-4xe^z)\,\mathbf{j}+0\,\mathbf{k}=\mathbf{0}$$

Since the domain of $\mathbf{F}$ is all of $\mathbb{R}^3$, which is simply connected, $\mathbf{F}$ is conservative, implying a potential function $f(x,y,z)$ exists such that $\mathbf{F}=\nabla f$. To find $f(x,y,z)$,

$$\frac{\partial f}{\partial x}=4xe^z \implies f(x,y,z)=\int 4xe^z\,dx=2x^2e^z+g(y,z)$$

$$\frac{\partial f}{\partial y}=\frac{\partial g}{\partial y}=\cos y \implies g(y,z)=\sin y+h(z) \implies f(x,y,z)=2x^2e^z+\sin y+h(z)$$

$$\frac{\partial f}{\partial z}=2x^2e^z+\frac{dh}{dz}=2x^2e^z \implies \frac{dh}{dz}=0 \implies h(z)=\text{constant (choose to be 0)}$$

$$f(x,y,z)=2x^2e^z+\sin y$$

We then have

$$\text{Work}=\int_{\mathcal{C}}\mathbf{F}\cdot d\mathbf{r}=\int_{\mathcal{C}}\nabla f\cdot d\mathbf{r}=f(x,y,z)\Big|_{(1,\frac{\pi}{2},1)}^{(\sqrt{3},\frac{3\pi}{2},9)}=2(3)e^9-1-\left[2(1)e^1+1\right]=6e^9-2e-2=2\left(3e^9-e-1\right)$$

(c) This curve is an ellipse, which is closed. Since the vector field is conservative,

$$\text{Work}=\oint_{\mathcal{C}}\mathbf{F}\cdot d\mathbf{r}=0$$

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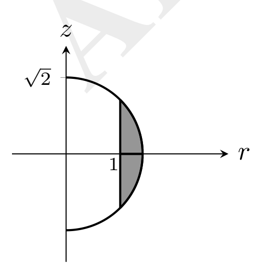
3. [APPM 2350 Exam (30 pts)] The following problems are not related.

(a) (18 pts) Use Stokes' Theorem to evaluate $\int_{\mathcal{C}} 2z\,dx+x\,dy+y^2\,dz$, where $\mathcal{C}$ is the trace of the surface $z=4-x^2-y^2$ in the xy-plane, oriented counterclockwise.

(b) (12 pts) The charge density in a solid metal ball with radius $\sqrt{2}$ feet is given by $q(\rho,\theta,\phi)=3\sin\left(\frac{\theta}{2}\right)$ Coulombs per cubic foot. Use spherical coordinates to find the total charge in the portion of the ball whose cross section for an arbitrary θ is shown in the following figure. Hint: $(\cot x)'=-\csc^2 x$



SOLUTION:

(a) From the given line integral, $\mathbf{F} = \langle 2z, x, y^2 \rangle$ and

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2z & x & y^2 \end{vmatrix} = 2y\,\mathbf{i} + 2\,\mathbf{j} + \mathbf{k}$$

The given orientation for $\mathcal{C}$ induces an upward pointing normal vector to the surface. Projecting the surface onto the xy-plane gives $\mathbf{p} = \mathbf{k}$ with region of integration $\mathcal{R}$ the disk of radius 2. For the given surface, $g(x, y, z) = x^2 + y^2 + z \implies \nabla g = \langle 2x, 2y, 1\rangle$. Then $|\nabla g \cdot \mathbf{p}| = 1$, we choose $+\nabla g$ for the normal vector to the surface and

$$\nabla\times\mathbf{F}\cdot\frac{+\nabla g}{|\nabla g\cdot\mathbf{p}|} = \langle 2y, 2, 1\rangle\cdot\langle 2x, 2y, 1\rangle = 4xy + 4y + 1$$

Thus,

$$\int_{\mathcal{C}} 2z\,\mathrm{d}x + x\,\mathrm{d}y + y^2\,\mathrm{d}z = \iint_{\mathcal{C}} \nabla\times\mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S = \iint_{x^2+y^2\le 4} (4xy + 4y + 1)\,\mathrm{d}A \qquad \text{switch to polar coordinates}$$

$$= \int_0^{2\pi}\int_0^2 [(4r\cos\theta)(r\sin\theta) + 4r\sin\theta + 1]\,r\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^{2\pi}\int_0^2 \left(2r^3\sin 2\theta + 4r^2\sin\theta + r\right)\mathrm{d}r\,\mathrm{d}\theta$$

$$= \int_0^{2\pi}\left(\frac{1}{2}r^4\sin 2\theta + \frac{4}{3}r^3\sin\theta + \frac{1}{2}r^2\right)\Bigg|_0^2\,\mathrm{d}\theta = \int_0^{2\pi}\left(8\sin 2\theta + \frac{32}{3}\sin\theta + 2\right)\mathrm{d}\theta$$

$$= \left(-4\cos 2\theta - \frac{32}{3}\cos\theta + 2\theta\right)\Bigg|_0^{2\pi} = 4\pi$$

Alternatively, the surface $g(x, y, z) = z = 0$ shares the boundary with the paraboloid. Then $\nabla g = \mathbf{k}$ and projecting onto the xy-plane gives $\mathbf{p} = \mathbf{k}$ with the disk of radius 2 centered at the origin as the region of integration. Given the orientation of $\mathcal{C}$ we choose $+\nabla g$ for the normal to the surface. Then

$$\nabla\times\mathbf{F}\cdot\frac{+\nabla g}{|\nabla g\cdot\mathbf{p}|} = \langle 2y, 2, 1\rangle\cdot\langle 0, 0, 1\rangle = 1$$

and

$$\int_{\mathcal{C}} 2z\,\mathrm{d}x + x\,\mathrm{d}y + y^2\,\mathrm{d}z = \iint_{\mathcal{C}} \nabla\times\mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S = \iint_{x^2+y^2\le 4} \mathrm{d}A = \text{area of circle of radius } 2 = 4\pi$$

(b) Letting $\mathcal{E}$ represent the region of interest,

$$\text{Charge} = \iiint_{\mathcal{E}} q\,\mathrm{d}V = \int_0^{2\pi}\int_{\pi/4}^{3\pi/4}\int_{\csc\phi}^{\sqrt{2}} 3\sin\left(\frac{\theta}{2}\right)\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$$

$$= \int_0^{2\pi}\int_{\pi/4}^{3\pi/4} \rho^3\Big|_{\csc\phi}^{\sqrt{2}}\sin\phi\sin\left(\frac{\theta}{2}\right)\mathrm{d}\phi\,\mathrm{d}\theta = \int_0^{2\pi}\int_{\pi/4}^{3\pi/4}\left(2\sqrt{2}\sin\phi - \csc^2\phi\right)\sin\left(\frac{\theta}{2}\right)\mathrm{d}\phi\,\mathrm{d}\theta$$

$$\int_0^{2\pi}\left(-2\sqrt{2}\cos\phi + \cot\phi\right)\Big|_{\pi/4}^{3\pi/4}\sin\left(\frac{\theta}{2}\right)\mathrm{d}\theta = \int_0^{2\pi} 2\sin\left(\frac{\theta}{2}\right)\mathrm{d}\theta = -4\cos\left(\frac{\theta}{2}\right)\Big|_0^{2\pi} = 8 \text{ Coulombs}$$

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4. [APPM 2350 Exam (32 pts)] Let $\mathcal{S}$ be the surface of the solid bounded by $z = 4 - y, z = 0, y = 0, x = 0, x = 6$ and let $\mathbf{F} = (x + 1)e^z\mathbf{i} + ye^z\mathbf{j} + e^z\mathbf{k}$. You need to compute the outward flux of $\mathbf{F}$ through $\mathcal{S}$.

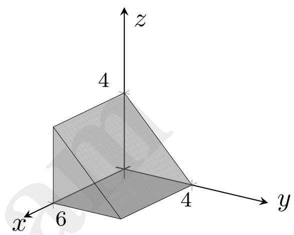
(a) (15 pts) Begin by computing the outward flux of $\mathbf{F}$ through the portion of the surface lying in the yz-plane.

(b) (2 pts) How many more calculations similar to the one in part (a) are required to find the flux? Do not compute it/them, just state how many.

(c) (15 pts) Rather than evaluating surface integrals to compute the flux, find the outward flux using an appropriate Calculus 3 theorem instead.

SOLUTION:

The solid is the wedge shown in the following figure.



(a) The surface, $\mathcal{S}_0$, is the plane $x = 0$ so that $g(x, y, z) = x \implies \nabla g = \mathbf{i}$. Project the surface onto the yz-plane so that $\mathbf{p} = \mathbf{i}$, $|\nabla g \cdot \mathbf{p}| = 1$ with the region of integration $0 \le y \le 4, 0 \le z \le 4 - y$. For the outward flux, choose $-\nabla g$ giving

$$\mathbf{F} \cdot \frac{-\nabla g}{|\nabla g \cdot \mathbf{p}|} = -(x+1)e^z = -e^z$$

when the surface is used to eliminate x (necessary since the integration region does involve x). Then

$$\text{Flux} = \iint_{\mathcal{S}_0} \mathbf{F} \cdot \mathbf{n}\, \mathrm{d}S = \int_0^4 \int_0^{4-y} -e^z \,\mathrm{d}z\,\mathrm{d}y = -\int_0^4 e^z \Big|_0^{4-y} \mathrm{d}y$$

$$= \int_0^4 \left(1 - e^{4-y}\right) \mathrm{d}y = \left(y + e^{4-y}\right)\Big|_0^4 = 5 - e^4$$

(b) 4 more surface integrals are required to compute the flux through $\mathcal{S}$: the parts of the wedge in the xy-plane, the xz-plane, the plane $x = 6$ and the plane $z = 4 - y$.

(c) Use Gauss' Divergence Theorem to compute the flux, with $\mathcal{E}$ the solid contained in $\mathcal{S}$.

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}\left[(x+1)e^z\right] + \frac{\partial}{\partial y}(ye^z) + \frac{\partial}{\partial z}(e^z) = 3e^z$$

$$\iint_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n}\,\mathrm{d}S = \iiint_{\mathcal{E}} \nabla \cdot \mathbf{F}\,\mathrm{d}V = \int_0^6 \int_0^4 \int_0^{4-y} 3e^z \,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

$$= \left(3 \int_0^6 \mathrm{d}x\right) \int_0^4 e^z \Big|_0^{4-y} \mathrm{d}y = 18 \int_0^4 \left(e^{4-y} - 1\right) \mathrm{d}y$$

$$= 18\left(-e^{4-y} - y\right)\Big|_0^4 = 18\left[-1 - 4 - \left(-e^4 - 0\right)\right] = 18\left(e^4 - 5\right)$$

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5. [APPM 2350 Exam (28 pts)] The elevation of the ground in a certain area is given by $f(x, y) = x^2 - y^2$.

(a) (2 pts) Identify the quadric surface given by the elevation function.

(b) (9 pts) You are standing on the surface at the point $P = (5, 10, -75)$ and decide to head in the direction $\mathbf{v} = 3\,\mathbf{i} + 4\mathbf{j}$.

i. (5 pts) At the instant you begin walking from P, determine if your elevation will be increasing or decreasing and find the rate of increase or decrease.

ii. (4 pts) From the point P, what is the largest (in absolute value) rate of change in elevation that you can attain?

(c) (5 pts) Use the chain rule to find the rate of change of elevation along the path $(x, y) = \left(t, \frac{1}{2}t^2 - 1\right)$ when $t = 2$.

(d) (12 pts) Suppose you decide to hike along a path whose (x, y) coordinates are constrained to satisfy $2y - x^2 = -2$. Find the value and location(s) of highest elevation you will reach.

SOLUTION:

(a) hyperbolic paraboloid

(b) $\nabla f = \langle 2x, -2y \rangle$

i.

$$\left.\frac{\mathrm{d}f}{\mathrm{d}s}\right|_{(5,10)} = D_{\mathbf{u}} f(5, 10) = \nabla f(5, 10) \cdot \frac{\mathbf{v}}{\|\mathbf{v}\|} = \langle 2(5), -2(10) \rangle \cdot \frac{1}{5}\langle 3, 4 \rangle = -10 \ \text{(elevation decreasing)}$$

ii.

$$\|\nabla f(5,10)\| = \|\langle 10, -20\rangle\| = \sqrt{10^2 + (-20)^2} = 10\sqrt{5}$$

(c)

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} = (2x)(1) + (-2y)(t) = 2t - 2t\left(\frac{1}{2}t^2 - 1\right) = 4t - t^3 \implies \left.\frac{\mathrm{d}f}{\mathrm{d}t}\right|_{t=2} = 0$$

(d) Use Lagrange Multipliers with objective function $f(x,y) = x^2 - y^2$ and constraint $g(x,y) = 2y - x^2 = -2$.

$$f_x = 2x \qquad g_x = -2x$$

$$f_y = -2y \qquad g_y = 2$$

$$2x = -2x\lambda \implies x(1+\lambda) = 0 \implies x = 0 \text{ or } \lambda = -1$$

$$-2y = 2\lambda \implies y = -\lambda$$

If $x = 0$, the constraint yields $y = -1$ so a critical point is $(0,-1)$ with $f(0,-1) = -1$. If $\lambda = -1$, then $y = 1$ and the constraint gives $2(1) - x^2 = -2 \implies x = \pm 2$ with $(\pm 2, 1)$ as critical points and $f(\pm 2, 1) = 3$. The highest elevation reached is 3, attained at $(\pm 2, 1)$.

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