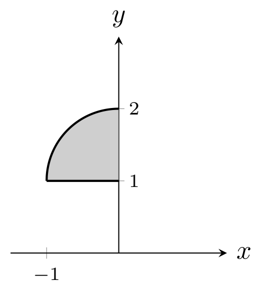
1. [APPM 2350 Exam (19 pts)] Sketch the region of integration and evaluate the following integral by switching to polar coordinates.

$$\int_1^2 \int_{-\sqrt{2y-y^2}}^0 \frac{y}{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y$$

Potentially helpful information: $\cos^2 x = \frac{1}{2}(1+\cos 2x)$ $\quad \sin^2 x = \frac{1}{2}(1-\cos 2x)$

SOLUTION:

The lower bound on the first integral gives $x = -\sqrt{2y-y^2} \implies x^2+y^2-2y=0 \implies x^2+(y-1)^2=1$ which is circle of radius 1 centered at $(0,1)$. Here is a sketch of the region.



The polar coordinates equation of the circle is $r = 2\sin\theta$ and that of the line is $r = \csc\theta$. The integrand in polar coordinates is $f(r,\theta) = \dfrac{\sin\theta}{r}$. Thus

$$\int_1^2 \int_{-\sqrt{2y-y^2}}^0 \frac{y}{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y = \int_{\pi/2}^{3\pi/4} \int_{\csc\theta}^{2\sin\theta} \sin\theta\,\mathrm{d}r\,\mathrm{d}\theta = \int_{\pi/2}^{3\pi/4} \sin\theta\, r\Big|_{\csc\theta}^{2\sin\theta}\mathrm{d}\theta$$

$$= \int_{\pi/2}^{3\pi/4} \left(2\sin^2\theta - 1\right)\mathrm{d}\theta = -\int_{\pi/2}^{3\pi/4} \cos 2\theta\,\mathrm{d}\theta = \frac{1}{2}\sin 2\theta\Big|_{3\pi/4}^{\pi/2} = \frac{1}{2}$$

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2. [APPM 2350 Exam (20 pts)] The density of pollen particles is given by $\rho(x,y) = 72(x+y)e^{x^2-y^2}$ g/cm^2. By making an appropriate change of variables, determine the mass of pollen contained in the rectangle, $\mathcal{R}$, enclosed by the lines $x-y=0$, $x-y=2$, $x+y=3$, $x+y=6$.

SOLUTION:

The region is bounded by two sets of parallel lines, suggesting the change of variables $u = x-y$ and $v = x+y$. This gives the new region of integration as $0 \le u \le 2$ and $3 \le v \le 6$. We have $x = \frac{1}{2}(u+v)$ and $y = \frac{1}{2}(v-u)$ so that

$$J(u,v) = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$$

$$\text{Mass} = \iint_{\mathcal{R}} 72(x+y)e^{x^2-y^2}\,\mathrm{d}A = 72\int_3^6 \int_0^2 v e^{uv}\left|\frac{1}{2}\right|\mathrm{d}u\,\mathrm{d}v$$

$$= 36\int_3^6 e^{uv}\Big|_0^2 \mathrm{d}v = 36\int_3^6 \left(e^{2v}-1\right)\mathrm{d}v = 36\left(\frac{1}{2}e^{2v}-v\right)\Big|_3^6$$

$$= 36\left[\frac{1}{2}e^{12} - 6 - \left(\frac{1}{2}e^6 - 3\right)\right] = 36\left(\frac{1}{2}e^{12} - \frac{1}{2}e^6 - 3\right) = 18\left(e^{12}-e^6-6\right)\ \text{g}$$

Integrating using the order $\mathrm{d}v\,\mathrm{d}u$ requires integration by parts and is not recommended. ■

3. [APPM 2350 Exam (21 pts)] Consider the region $\mathcal{W}$ below the fourth quadrant and inside the sphere $x^2+y^2+z^2=36$ between the planes $z=-3$ and $z=-3\sqrt{3}$. We want to find $B = \iiint_{\mathcal{W}} xyz\,\mathrm{d}V$.

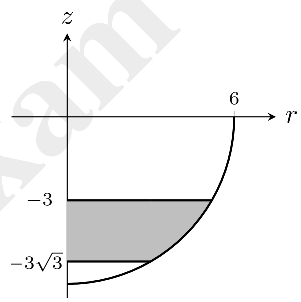
(a) Set up, but **DO NOT EVALUATE** the integral(s) necessary to compute B in rectangular/Cartesian coordinates using the order $\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$.

(b) Set up, but **DO NOT EVALUATE** the integral(s) necessary to compute B in cylindrical coordinates using the order $\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}\theta$.

(c) Set up, but **DO NOT EVALUATE** the integral(s) necessary to compute B in spherical coordinates using the order $\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$.

SOLUTION:

A sketch of the region of integration in the zr-plane (constant θ) follows.



(a)

$$B = \int_0^3 \int_{-\sqrt{9-x^2}}^0 \int_{-3\sqrt{3}}^{-3} xyz\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x + \int_0^3 \int_{-\sqrt{27-x^2}}^{-\sqrt{9-x^2}} \int_{-\sqrt{36-x^2-y^2}}^{-3} xyz\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$
$$+ \int_3^{3\sqrt{3}} \int_{-\sqrt{27-x^2}}^0 \int_{-\sqrt{36-x^2-y^2}}^{-3} xyz\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

Alternatively,

$$B = \int_0^3 \int_{-\sqrt{9-x^2}}^0 \int_{-3\sqrt{3}}^{-3} xyz\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x + \int_0^{3\sqrt{3}} \int_{-\sqrt{27-x^2}}^0 \int_{-\sqrt{36-x^2-y^2}}^{-3} xyz\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$
$$- \int_0^3 \int_{-\sqrt{9-x^2}}^0 \int_{-\sqrt{36-x^2-y^2}}^{-3} xyz\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

Another alternative:

$$B = \int_0^{3\sqrt{3}} \int_{-\sqrt{27-x^2}}^0 \int_{-\sqrt{36-x^2-y^2}}^{-3} xyz\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x - \int_0^3 \int_{-\sqrt{9-x^2}}^0 \int_{-\sqrt{36-x^2-y^2}}^{-3\sqrt{3}} xyz\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

(b)

$$B = \int_{3\pi/2}^{2\pi} \int_{-3\sqrt{3}}^{-3} \int_0^{\sqrt{36-z^2}} r^3 z\cos\theta\sin\theta\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}\theta$$

θ bounds can alternatively be $-\pi/2$ to 0.

(c)

$$B = \int_{3\pi/2}^{2\pi} \int_{2\pi/3}^{5\pi/6} \int_{-3\sec\phi}^{6} \rho^5 \sin^3\phi\cos\phi\cos\theta\sin\theta\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta + \int_{3\pi/2}^{2\pi} \int_{5\pi/6}^{\pi} \int_{-3\sec\phi}^{-3\sqrt{3}\sec\phi} \rho^5 \sin^3\phi\cos\phi\cos\theta\sin\theta\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$$

θ bounds can alternatively be $-\pi/2$ to 0.

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4. [APPM 2350 Exam (20 pts)] Evaluate $\iint_{\mathcal{S}} 48\sqrt{3}\,yz\,\mathrm{d}S$ where $\mathcal{S}$ is the portion of the surface

$$\sqrt{3}x = y + 2z^2 \;\text{ with }\; -\sqrt{3}/2 \le y \le 0, -y \le z \le \sqrt{3}/2$$

SOLUTION:

$$g(x,y,z) = \sqrt{3}x - y - 2z^2 \implies \nabla g = \left\langle \sqrt{3}, -1, -4z \right\rangle \implies \|\nabla g\| = \sqrt{3+1+16z^2} = 2\sqrt{1+4z^2}$$

The region of integration is in the yz-plane so that $\mathbf{p} = \mathbf{i}$ and $|\nabla g \cdot \mathbf{p}| = \sqrt{3}$.

$$\iint_{\mathcal{S}} 48\sqrt{3}\, yz \,\mathrm{d}S = 48\sqrt{3}\int_0^{\sqrt{3}/2}\int_{-z}^{0} yz\frac{2\sqrt{1+4z^2}}{\sqrt{3}}\,\mathrm{d}y\,\mathrm{d}z = 48\int_0^{\sqrt{3}/2} y^2\Big|_{-z}^{0} z\sqrt{1+4z^2}\,\mathrm{d}z$$

$$= -48\int_0^{\sqrt{3}/2} z^3\sqrt{1+4z^2}\,\mathrm{d}z \overset{u=1+4z^2}{=} -\frac{3}{2}\int_1^4\left(u^{3/2}-u^{1/2}\right)\mathrm{d}u = -\frac{3}{2}\left(\frac{2}{5}u^{5/2}-\frac{2}{3}u^{3/2}\right)\Big|_1^4$$

$$= \left(u^{3/2}-\frac{3}{5}u^{5/2}\right)\Big|_1^4 = \left[8-\frac{3}{5}(32)\right]-\left(1-\frac{3}{5}\right) = 7-\frac{93}{5} = -\frac{58}{5}$$

Alternatively,

$$\iint_{\mathcal{S}} 48\sqrt{3}\, yz\,\mathrm{d}S = 48\sqrt{3}\int_{-\sqrt{3}/2}^{0}\int_{-y}^{\sqrt{3}/2} yz\frac{2\sqrt{1+4z^2}}{\sqrt{3}}\,\mathrm{d}z\,\mathrm{d}y \overset{u=1+4z^2}{=} 12\int_{-\sqrt{3}/2}^{0}\int_{1+4y^2}^{4} yu^{1/2}\,\mathrm{d}u\,\mathrm{d}y$$

$$= 8\int_{-\sqrt{3}/2}^{0} y\left[8-\left(1+4y^2\right)^{3/2}\right]\mathrm{d}y = 64\int_{-\sqrt{3}/2}^{0} y\,\mathrm{d}y - 8\int_{-\sqrt{3}/2}^{0} y\left(1+4y^2\right)^{3/2}\mathrm{d}y$$

$$\overset{u=1+4y^2}{=} -24-\int_4^1 u^{3/2}\,\mathrm{d}u = -24-\frac{2}{5}u^{5/2}\Big|_4^1 = -24-\frac{2}{5}(1-32) = -\frac{58}{5}$$

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5. [APPM 2350 Exam (20 pts)] The electric charge q at a point (x, y, z) in space is equal to the square of the distance from the point to the origin. Find the average value of the charge on a wire that lies along the curve $\mathcal{C} = \left(\sin\left(\pi t^2\right), \sqrt{3}\pi t^2, \cos\left(\pi t^2\right)\right)$, $t > 0$ between the points $\left(0, \sqrt{3}\pi, -1\right)$ and $\left(0, 4\sqrt{3}\pi, 1\right)$.

SOLUTION:

The charge is given by $q(x, y, z) = x^2 + y^2 + z^2$. From the description of the path we have

$$\mathbf{r}(t) = \left\langle \sin\left(\pi t^2\right), \sqrt{3}\pi t^2, \cos\left(\pi t^2\right)\right\rangle \quad 1 \le t \le 2$$

$$\mathbf{r}'(t) = \left\langle 2\pi t\cos\left(\pi t^2\right), 2\sqrt{3}\pi t, -2\pi t\sin\left(\pi t^2\right)\right\rangle$$

$$\|\mathrm{r}'(t)\| = \sqrt{4\pi^2t^2\cos^2\left(\pi t^2\right)+12\pi^2t^2+4\pi^2t^2\sin^2\left(\pi t^2\right)} = 4\pi t \quad \text{since } t > 0$$

$$q(\mathbf{r}(t)) = \sin^2\left(\pi t^2\right)+3\pi^2t^4+\cos^2\left(\pi t^2\right) = 1+3\pi^2t^4$$

The average value of the charge is $q_{\text{avg}} = \dfrac{\int_{\mathcal{C}} q\,\mathrm{d}s}{\int_{\mathcal{C}}\mathrm{d}s}$.

$$\int_{\mathcal{C}} q\,\mathrm{d}s = \int_1^2\left(1+3\pi^2t^4\right)(4\pi t)\,\mathrm{d}t = \int_1^2\left(4\pi t+12\pi^3t^5\right)\mathrm{d}t$$

$$= \left(2\pi t^2+2\pi^3t^6\right)\Big|_1^2 = 8\pi+128\pi^3-\left(2\pi+2\pi^3\right)$$

$$= 6\pi+126\pi^3 = 6\pi\left(1+21\pi^2\right)$$

$$\int_{\mathcal{C}}\mathrm{d}s = \int_1^2 4\pi t\,\mathrm{d}t = 6\pi$$

$$\implies q_{\text{avg}} = \frac{6\pi\left(1+21\pi^2\right)}{6\pi} = 1+21\pi^2$$

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