1. [APPM 2350 Exam (22 pts)] The following problems are not related.

   (a) (10 pts) The critical points of the function $f(x, y) = x^3 - 12xy + 8y^3$ are $(0, 0)$ and $(2, 1)$. Classify each of these as either a saddle point or local extremum. Justify your answer mathematically.

   (b) (12 pts) Kalkthree Regional Park consists of the boundary and interior of the triangle described by $x = 0$, $y = 0$, and $x + y = 4$. If the elevation of the park is given by $f(x, y) = x^2 + 2xy - y^2 - 4x$, find the highest and lowest points in the park as well as their locations.

SOLUTION:

(a)

$$f_x(x,y) = 3x^2 - 12y \qquad f_{xx}(x,y) = 6x$$
$$f_y(x,y) = -12x + 24y^2 \qquad f_{yy}(x,y) = 48y$$
$$f_{xy} = -12$$

$$D(x,y) = f_{xx}(x,y)f_{yy}(x,y) - [f_{xy}(x,y)]^2 = (6x)(48y) - (-12)^2 = 144(2xy - 1)$$

$$D(0,0) = -144 < 0 \implies (0,0) \text{ is a saddle point}$$

$$D(2,1) = 432 > 0, f_{xx}(2,1) = 12 > 0 \implies f(2,1) = -8 \text{ is a local minimum}$$

(b) This is an application of the Extreme Value Theorem. We know that a maximum and minimum elevation will exist since $f(x, y)$ is continuous and the park occupies a closed, bounded region. Begin by finding the critical points:

$$f_x = 2x + 2y - 4 = 0 \implies x + y = 2$$
$$f_y = 2x - 2y = 0 \implies x - y = 0$$

The only solution to this system, and thus the only critical point, is $(x, y) = (1, 1)$ with $f(1, 1) = \boxed{-2}$.

Now check the boundaries. On the left boundary where $x = 0$, the elevation function is $f(0, y) = -y^2$ with $0 \le y \le 4$. This has maximum value of 0 at $y = 0$ and minimum value of -16 at $y = 4$. Thus $f(0, 0) = \boxed{0}$ and $f(0, 4) = \boxed{-16}$.

On the bottom boundary where $y = 0$, the elevation function is $f(x, 0) = x^2 - 4x$ with $0 \le x \le 4$. This is a upward opening parabola with x–intercepts and maximum value of 0 on the given interval at $x = 0$ and $x = 4$. The minimum value is -4 at $x = 2$. On the bottom boundary then $f(2, 0) = \boxed{-4}$ and $f(4, 0) = \boxed{0}$.

On the oblique boundary we have $f(x, 4 - x) = g(x) = -2x^2 + 12x - 16$ with $0 \le x \le 4$. Using the Extreme Value Theorem from single variable calculus, the critical point of this function of one variable is $g'(x) = -4x + 12 = 0 \implies x = 3$ so that $g(3) = 2$. At the endpoints, $g(0) = -16$ and $g(4) = 0$. When $x = 3$ on this line, $y = 1$ so that $f(3, 1) = \boxed{2}$. (Note, $g(0)$ and $g(4)$ correspond to points that have already been checked.)

The largest and smallest of the boxed numbers give the extreme values we seek. Maximum elevation is 2 at $(3, 1)$ and the minimum elevation is -16 at $(0, 4)$.

Note that Lagrange Multipliers can be used on the three boundaries, but requires more work.

2. [APPM 2350 Exam (20 pts)] You are contemplating building a house in the shape of a box (without a roof for now) to provide a quiet spot to do Calculus 3 homework. The house is to have a volume of 81 m^3. The wood for the bottom of the house costs 6 times as much (per unit area) as the wood for the sides. Using Lagrange Multipliers, find the dimensions of the house that will minimize the cost of the lumber.

SOLUTION:

Let the box have sides x, y, and z. The total cost of materials is $C(x, y, z) = 6xy + 2xz + 2yz$ (objective function to be minimized) and the constraint is $g(x, y, z) = xyz = 81$. Then $\nabla C = \lambda \nabla g$ and the constraint gives the following system of equations

$$6y + 2z = \lambda yz \qquad (1)$$
$$6x + 2z = \lambda xz \qquad (2)$$
$$2x + 2y = \lambda xy \qquad (3)$$
$$xyz = 81 \qquad (4)$$

From (4), none of the variables can be 0. Thus, solving (1) and (2) for λ yields

$$\frac{6y+2z}{yz} = \lambda = \frac{6x+2z}{xz} \implies \frac{6}{z}+\frac{2}{y} = \frac{6}{z}+\frac{2}{x} \implies x = y$$

Combining (2) and (3) and using the fact that $x = y$ gives

$$\frac{6x+2z}{xz} = \lambda = \frac{2x+2y}{xy} \implies \frac{6}{z}+\frac{2}{x} = \frac{2}{y}+\frac{2}{x} = \frac{2}{x}+\frac{2}{x} \implies z = 3x$$

Equation (4) then gives $x(x)(3x) = 81 \Rightarrow x = 3$. To minimize building costs, the dimensions of the house should be $3\,\text{m} \times 3\,\text{m} \times 9\,\text{m}$.
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3. [APPM 2350 Exam (34 pts)] The following problems are not related.
   (a) (8 pts) The ideal gas law for a particular gas is given by $T = PV$, where T is temperature, P is pressure, and V is volume. You have an indestructible balloon of volume 1 m^3 at a temperature of 300 Kelvin filled with this gas. You want to test the limits of the indestructible balloon so you place it in an oven. The temperature of the gas is immediately increasing at a rate of 5 Kelvin per second. The volume is increasing at a rate of 0.01 cubic meters per second. At this instant, how fast is the pressure changing?
   (b) (8 pts) If $z = x + y + \sqrt{xy}$ and $x = te^s$, $y = s^2 + t^2$, find z_t when $s = 0$ and $t = 4$.
   (c) (18 pts) Let $f(x, y) = \ln(2x + y)$.
      i. (4 pts) Find and sketch the domain of $f(x, y)$.
      ii. (4 pts) Find and sketch the level curve corresponding to $f(x, y) = 2$. Label all, if any, intercepts.
      iii. (2 pts) Evaluate $\displaystyle\lim_{(x,y)\to(e,e)} f(x, y)$.
      iv. (8 pts) Find the second order Taylor polynomial for $f(x, y)$ centered at $(1, 2)$. Simplify your answer.

SOLUTION:

(a) We can rewrite the ideal gas law as $P(t) = T(t)V^{-1}(t)$. The chain rule then gives

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{\partial P}{\partial T}\frac{\mathrm{d}T}{\mathrm{d}t} + \frac{\partial P}{\partial V}\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{1}{V}\frac{\mathrm{d}T}{\mathrm{d}t} - \frac{T}{V^2}\frac{\mathrm{d}V}{\mathrm{d}t}$$

so that at the instant the balloon is placed in the oven we have

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \left(\frac{1}{1\,\text{m}^3}\right)\left(5\frac{\text{K}}{\text{sec}}\right) - \left(\frac{300}{1}\frac{\text{K}}{\text{m}^6}\right)\left(\frac{1}{100}\frac{\text{m}^3}{\text{sec}}\right) = 2\frac{\text{K}}{\text{m}^3\text{-sec}}$$

Alternatively, using the equation as given, $T(t) = P(t)V(t)$, we have

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{\partial T}{\partial p}\frac{\mathrm{d}P}{\mathrm{d}t} + \frac{\partial T}{\partial V}\frac{\mathrm{d}V}{\mathrm{d}t} = V\frac{\mathrm{d}P}{\mathrm{d}t} + P\frac{\mathrm{d}V}{\mathrm{d}t} \implies \frac{\mathrm{d}P}{\mathrm{d}t} = \frac{1}{V}\left(\frac{\mathrm{d}T}{\mathrm{d}t} - \frac{T}{V}\frac{\mathrm{d}V}{\mathrm{d}t}\right)$$

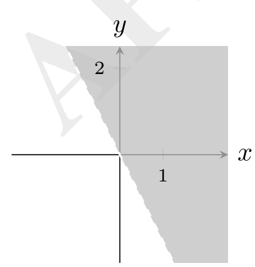
(b)

$$z_t = z_x x_t + z_y y_t = \left(1 + \frac{1}{2}x^{-1/2}y^{1/2}\right)e^s + \left(1 + \frac{1}{2}x^{1/2}y^{-1/2}\right)(2t)$$

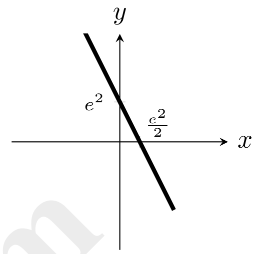
At $s = 0$ and $t = 4$, $x = 4$ and $y = 16$. Thus

$$z_t = \left[1 + \frac{1}{2}\left(4^{-1/2}\right)\left(16^{1/2}\right)\right](1) + \left[1 + \frac{1}{2}\left(4^{1/2}\right)\left(16^{-1/2}\right)\right](8) = 12$$

(c) i. We need $2x + y > 0$ or $y > -2x$.



ii. $\ln(2x + y) = 2 \implies 2x + y = e^2 \implies y = -2x + e^2$



iii. Since $f(x,y)$ is continuous on its domain and (e,e) is in its domain, use direct substitution to get

$$\lim_{(x,y)\to(e,e)} \ln(2x+y) = \ln(2e+e) = \ln(3e) = 1 + \ln 3$$

iv. $f(1,2) = \ln 4$

$$f_x(x,y) = \frac{2}{2x+y} \implies f_x(1,2) = \frac{1}{2} \qquad f_y(x,y) = \frac{1}{2x+y} \implies f_y(1,2) = \frac{1}{4}$$

$$f_{xx}(x,y) = -\frac{4}{(2x+y)^2} \implies f_{xx}(1,2) = -\frac{1}{4} \qquad f_{yy} = -\frac{1}{(2x+y)^2} \implies f_{yy}(1,2) = -\frac{1}{16}$$

$$f_{xy}(x,y) = -\frac{2}{(2x+y)^2} \implies f_{xy}(1,2) = -\frac{1}{8}$$

Therefore, the second degree Taylor polynomial is

$$\begin{aligned}
T_2(x,y) &= f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2) \\
&\quad + \frac{1}{2!}\left[f_{xx}(1,2)(x-1)^2 + 2f_{xy}(1,2)(x-1)(y-2) + f_{yy}(1,2)(y-2)^2\right] \\
&= \ln 4 + \frac{1}{2}(x-1) + \frac{1}{4}(y-2) + \frac{1}{2}\left[-\frac{1}{4}(x-1)^2 + 2\left(-\frac{1}{8}\right)(x-1)(y-2) - \frac{1}{16}(y-2)^2\right] \\
&= \ln 4 + \frac{1}{2}(x-1) + \frac{1}{4}(y-2) - \frac{1}{8}(x-1)^2 - \frac{1}{8}(x-1)(y-2) - \frac{1}{32}(y-2)^2
\end{aligned}$$

4. [APPM 2350 Exam (24 pts)] The following problems are not related.

(a) (5 pts) Given the surface $3x^2 + y^2 - z^2 = -5$, find the equation of the tangent plane at the point $(1,1,-3)$. Write your simplified answer in the form $ax + by + cz = d$.

(b) (19 pts) The depth of a lake is given by the equation $h(x,y) = 2x^4 + 3y^2 - 10$. You and some friends are in a boat on the surface of the lake, which is located in the xy–plane. All length/depth units are in meters.

i. (8 pts) At the point $(1,-1,0)$ you are heading in the direction $2\mathbf{i} - \mathbf{j}$. Is the lake getting shallower or deeper? At what rate?

ii. (8 pts) From the point $(1,-1,0)$, in what direction(s) should you set sail so as to follow the level curve through the point? Write your answer as a unit vector(s).

iii. (3 pts) Your friends are really in a hurry to get to deeper water. They ask you to set sail from the point $(1,-1,0)$ in a direction that will make the depth of the lake change at a rate of 12 m/m. What should you tell them? Justify your answer mathematically.

SOLUTION:

(a) Let $F(x,y,z) = 3x^2 + y^2 - z^2$. Then we are finding the tangent plane to the level surface $F(x,y,z) = -5$. The equation of the tangent plane is given by

$$F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0$$

$$F_x(x,y,z) = 6x \qquad F_y(x,y,z) = 2y \qquad F_z(x,y,z) = -2z$$

$$6(1)(x-1) + 2(1)(y-1) - 2(-3)(z+3) = 0$$

$$6x - 6 + 2y - 2 + 6z + 18 = 0$$

$$3x + y + 3z = -5$$

(b) i. Need to find the directional derivative. A unit vector in the direction of interest is $\mathbf{u} = \dfrac{2}{\sqrt{5}}\,\mathbf{i} - \dfrac{1}{\sqrt{5}}\,\mathbf{j}$. The gradient of h is $\nabla h = 8x^3\,\mathbf{i} + 6y\,\mathbf{j}$. Thus

$$\left.\frac{\mathrm{d}h}{\mathrm{d}s}\right|_{(1,-1)} = D_{\mathbf{u}}h(1,-1) = \nabla h(1,-1)\cdot\mathbf{u} = \langle 8,-6\rangle \cdot \left\langle \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \right\rangle = \frac{22}{\sqrt{5}} = \frac{22\sqrt{5}}{5}$$

Since $h(1,-1) = -5$ and $D_{\mathbf{u}}h(1,-1) > 0$, the lake is getting shallower at a rate of $\frac{22\sqrt{5}}{5}$ m/m.

ii. Following a level curve results in a vanishing directional derivative. Let the direction we seek be given by $\mathbf{u} = \langle u_1, u_2 \rangle$. Then

$$\left.\frac{\mathrm{d}h}{\mathrm{d}s}\right|_{(1,-1)} = D_{\mathbf{u}}h(1,-1) = \nabla h(1,-1)\cdot\mathbf{u} = \langle 8,-6\rangle \cdot \langle u_1, u_2\rangle = 0 \implies 8u_1 - 6u_2 = 0 \implies u_1 = \frac{3}{4}u_2$$

Choosing $u_2 = 4$ gives $u_1 = 3$ giving $\mathbf{u} = \langle 3, 4\rangle$. Letting $u_2 = -4$ gives $u_1 = -3$ gives $\mathbf{u} = \langle -3, -4\rangle$. Setting sail in the directions $\left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$ or $\left\langle -\frac{3}{5}, -\frac{4}{5} \right\rangle$ will follow the level curve through the point $(1,-1)$.

iii. The greatest rate at which the depth can increase at $(1,-1)$ is $\|\nabla h(1,-1)\| = \sqrt{8^2 + (-6)^2} = 10$. You should tell your friends that you cannot find a direction to set sail from the point that will make the depth increase at a rate of 12 m/m.

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