1. [APPM 2350 Exam (19 pts)] A bee leaves its hive located at the point $A(1, 2, 3)$ and flies in a straight line to a sunflower located at point $B(-1, 4, 7)$. From there it flies in a straight line to a daisy at point $C(2, -5, -1)$ and returns on a straight path back to the hive.

   (a) (2 pts) How far is the sunflower from the hive?

   (b) (5 pts) Find the area of the triangular region inside the bee's path.

   (c) (6 pts) Find the equation of the plane containing the bee's path. Write your answer in the form $ax + by + cz = d$.

   (d) (3 pts) The queen bee leaves the hive and crawls along a straight branch to point D at the origin. Find the volume of the parallelepiped formed by the queen bee's path, and the bee's path from the hive to the sunflower and from the hive to the daisy.

   (e) (3 pts) Is the angle between the queen bee's path and the path from the sunflower to the daisy acute or obtuse (greater than $\pi/2$ radians)? Justify your answer.

SOLUTION:

(a) The vector giving the path from the hive to the sunflower is $\overrightarrow{AB} = \langle -2, 2, 4\rangle$. Its magnitude gives the distance from the hive the sunflower as $\|\overrightarrow{AB}\| = \sqrt{(-2)^2 + 2^2 + 4^2} = \sqrt{24} = 2\sqrt{6}$

(b) The vector from the hive to the daisy is $\overrightarrow{AC} = \langle 1, -7, -4\rangle$. This vector and $\overrightarrow{AB}$ form a parallelogram. The area of the triangular region inside the bee's path is one-half the area of this parallelogram, that is,

$$\text{Area} = \frac{1}{2}\left\|\overrightarrow{AB} \times \overrightarrow{AC}\right\| = \frac{1}{2}\left\| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 2 & 4 \\ 1 & -7 & -4 \end{vmatrix} \right\| = \frac{1}{2}\|20\,\mathbf{i} - 4\,\mathbf{j} + 12\,\mathbf{k}\| = \frac{\sqrt{560}}{2} = 2\sqrt{35}$$

Note that you can use any two of the vectors that make up the bee's path to find the requested area.

(c) To find the equation of a plane we need a normal vector and a point in the plane. We already have a normal from part (b) as $\mathbf{n} = 20\,\mathbf{i} - 4\,\mathbf{j} + 12\,\mathbf{k}$. Using point A as a point in the plane gives

$$20(x-1) - 4(y-2) + 12(z-3) = 0 \implies 20x - 4y + 12z = 48 \ \text{ or } \ 5x - y + 3z = 12$$

Again, note that any of the vectors making up the bee's path and any of the points the bee visited could be used in the above calculation.

(d) The vector giving the queen bee's path is $\overrightarrow{AD} = \langle -1, -2, -3\rangle$ and we have the other two vectors from early. The volume is given by the absolute value of the scalar triple product of the three vectors (taken in any order) as

$$\text{Volume} = \left|\overrightarrow{AD} \cdot \left(\overrightarrow{AB} \times \overrightarrow{AC}\right)\right| = \left| \begin{vmatrix} -1 & -2 & -3 \\ -2 & 2 & 4 \\ 1 & -7 & -4 \end{vmatrix} \right| = |-48| = 48$$

Alternatively, using previous computations,

$$\text{Volume} = \left|\overrightarrow{AD} \cdot \left(\overrightarrow{AB} \times \overrightarrow{AC}\right)\right| = |\langle -1, -2, -3\rangle \cdot \langle 20, -4, 12\rangle| = |-48| = 48$$

(e) The vector from the sunflower to the daisy is $\overrightarrow{BC} = \langle 3, -9, -8\rangle$. If the dot product between two vectors is positive, the angle between them is acute; it is obtuse otherwise.

$$\overrightarrow{AD} \cdot \overrightarrow{BC} = \langle -1, -2, -3\rangle \cdot \langle 3, -9, -8\rangle = 39 > 0 \implies \text{angle between the vectors is acute}$$

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2. [APPM 2350 Exam (26 pts)] A particle moves along the path parameterized by $\mathbf{r}(t) = 2t^2\,\mathbf{i} + 6\,\mathbf{j} + \frac{4}{3}t^3\,\mathbf{k}$, $t \geq 0$.

   (a) (13 pts) When $t = \sqrt{3}$ find the following. Simplify your answers.

      i. (3 pts) Where is the particle?

      ii. (5 pts) How fast is the particle moving?

      iii. (5 pts) Find a unit vector in the direction of motion of the particle.

   (b) (3 pts) When does the particle reach the point $(18, 6, 36)$?

   (c) (8 pts) How far did the particle travel moving from the point with position vector $6\,\mathbf{j}$ to the point where $t = \sqrt{2}$.

   (d) (2 pts) Find the binormal vector $\mathbf{B}$. Hint: Do not spend time computing $\mathbf{T}$ and $\mathbf{N}$. Instead, just visualize the path.

SOLUTION:

(a) i.
$$\mathbf{r}(\sqrt{3}) = 2(\sqrt{3})^2\,\mathbf{i} + 6\,\mathbf{j} + \frac{4}{3}(\sqrt{3})^3\mathbf{k} = 6\,\mathbf{i} + 6\,\mathbf{j} + 4\sqrt{3}\,\mathbf{k}$$

ii.
$$\mathbf{r}'(t) = 4t\,\mathbf{i} + 0\,\mathbf{j} + 4t^2\,\mathbf{k}$$
$$\|\mathbf{r}'(t)\| = \sqrt{16t^2 + 16t^4} = \sqrt{16t^2\,(1+t^2)} = 4|t|\sqrt{1+t^2} = 4t\sqrt{1+t^2} \text{ since } t \geq 0$$

The speed of the particle is given by
$$\|\mathbf{r}'(\sqrt{3})\| = 4\sqrt{3}\sqrt{1+(\sqrt{3})^2} = 8\sqrt{3}$$

iii. This is asking for $\mathbf{T}(\sqrt{3})$.
$$\mathbf{T}(\sqrt{3}) = \frac{\mathbf{r}'(\sqrt{3})}{\left\|\mathbf{r}'(\sqrt{3})\right\|} = \frac{4\sqrt{3}\,\mathbf{i} + 0\,\mathbf{j} + 12\,\mathbf{k}}{8\sqrt{3}} = \frac{1}{2}\,\mathbf{i} + \frac{\sqrt{3}}{2}\,\mathbf{k}$$

(b) We need $2t^2 = 18$ and $\frac{4}{3}t^3 = 36$. With $t \geq 0$ these yield $t = 3$.

(c) The particle has position vector $\mathbf{r}(t) = 6\,\mathbf{j}$ when $t = 0$. So the requested travel distance is

$$s = \int_0^{\sqrt{2}} 4t\sqrt{1+t^2}\,\mathrm{d}t \overset{u=1+t^2}{=} 2\int_1^3 \sqrt{u}\,\mathrm{d}u = \frac{4}{3}\left(3\sqrt{3}-1\right)$$

(d) The osculating plane, containing $\mathbf{T}$ and $\mathbf{N}$, is $y = 6$ for all time. Since $\mathbf{B}$ is a unit vector normal to the osculating plane (and equal to $\mathbf{T}\times\mathbf{N}$), we have $\mathbf{B} = \pm\mathbf{j}$. Viewing the path looking in the positive y direction shows a "parabola-like" curve $\left(z = \frac{\sqrt{2}}{3}x^{3/2}\right)$ opening upward, indicating that the curve is turning to the left, implying that $\mathbf{N}$ points to the left forcing $\mathbf{B} = -\mathbf{j}$.

3. [APPM 2350 Exam (28 pts)] The following problems are not related.

(a) (8 pts) Find the equation of the surface consisting of all points that are equidistant from the point $(-1, 1, 2)$ and the plane $y = -1$. Is this one of the quadric surfaces? If so, identify it.

(b) (12 pts) Consider the equation $x^2 + z^2 - 4x - 2y - 2z = y^2 - c$.

i. (5 pts) For what value of c will the equation describe a cone?

ii. (3 pts) For the value of c found in part (i), find the vertex of the cone.

iii. (4 pts) For the value of c found in part (i), find the trace of the surface in the plane $y = 2$, giving its equation and a geometrical description in words.

(c) (8 pts) Consider the plane passing through the point $(-1, 1, -14)$ with normal vector $-\mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}$. Find the point where the line with symmetric equations $\dfrac{x-1}{-2} = \dfrac{y+2}{3} = z + 4$ intersects the plane.

SOLUTION:

(a) Let (x, y, z) be an arbitrary point on the surface. Then the distance from this point to the plane is $|y+1|$ and from this point to the given point is $\sqrt{(x+1)^2 + (y-1)^2 + (z-2)^2}$. Equating this two distances and simplifying yields

$$|y+1| = \sqrt{(x+1)^2 + (y-1)^2 + (z-2)^2}$$
$$(y+1)^2 = (x+1)^2 + (y-1)^2 + (z-2)^2$$
$$y^2 + 2y + 1 - (y^2 - 2y + 1) = (x+1)^2 + (z-2)^2$$
$$y = \frac{1}{4}\left[(x+1)^2 + (z-2)^2\right] \quad \text{(circular) paraboloid}$$

(b) i. Complete the square to get

$$x^2 - 4x + 4 - 4 - (y^2 + 2y + 1 - 1) + z^2 - 2z + 1 - 1 = -c$$
$$(x-2)^2 - (y+1)^2 + (z-1)^2 = -c + 4 - 1 + 1 = 4 - c$$

For this to describe a cone we need $c = 4$.

ii. The quadric surface is $(x-2)^2 - (y+1)^2 + (z-1)^2 = 0$ so the vertex of the cone is $(2, -1, 1)$.

iii. Set $y = 2$ to yield

$$(x-2)^2 - 9 + (z-1)^2 = 0 \implies (x-2)^2 + (z-1)^2 = 9$$

which is a circle of radius 3 centered at $(2, 2, 1)$.

(c) The equation of the plane is $-1(x+1) + 2(y-1) + 3(z+14) = 0$. The parametric equations of the given line are $x = 1 - 2t$, $y = -2 + 3t$, and $z = -4 + t$. The line intersects the plane if its coordinates satisfy the equation of the plane:

$$-1(1 - 2t + 1) + 2(-2 + 3t - 1) + 3(-4 + t + 14) = 0 \implies 11t = -22 \implies t = -2$$

Using this value of t in the parametric equations of the line yield the point of intersection as $(5, -8, -6)$.

4. [APPM 2350 Exam (27 pts)] The following problems are not related.

(a) (9 pts) Let $\mathbf{r}(t) = \left\langle \sqrt{t+1}, \cos t, t^4 - 8t^2 \right\rangle,\ t > -1$.

i. (6 pts) Find all values of t where the path is parallel to the xy-plane.

ii. (3 pts) Is the path ever parallel to the yz-plane? Justify your answer.

(b) (8 pts) An object moving with velocity $\mathbf{v}(t) = (t+2)\,\mathbf{i} + t^2\,\mathbf{j} + e^{-t/3}\mathbf{k}$ passes through the point $(4, 0, -2)$ at time $t = 0$. Find the vector-valued function describing the position of the object at any time t.

(c) (10 pts) Let $\mathbf{r}(t) = (1+t)\,\mathbf{i} + \left(t^2 - 2t\right)\mathbf{j} + 0\,\mathbf{k},\ -\infty < t < \infty$.

i. (5 pts) Find the curvature of the path when $t = 1$.

ii. (5 pts) Find the point(s) on the path where the acceleration vector consists of only a normal component, and compute the magnitude of the normal component of the acceleration at that(those) point(s).

SOLUTION:

(a) i. The tangent vector to the path is

$$\mathbf{r}'(t) = \left\langle \frac{1}{2\sqrt{t+1}}, -\sin t, 4t^3 - 16t \right\rangle$$

To be parallel to the xy-plane, the $\mathbf{k}$-component of the tangent vector must vanish. This occurs if

$$4t^3 - 16t = 4t(t^2 - 4) = 4t(t-2)(t+2) = 0 \implies t = -2, 0, 2$$

Since we are only considering $t > -1$ the path is parallel to the xy-plane for $t = 0$ and $t = 2$.

ii. To be parallel to the yz-plane, the $\mathbf{i}$-component of the tangent vector must be zero. Since $\dfrac{1}{2\sqrt{t+1}}$ is never 0, the path is never parallel to the yz-plane.

(b)

$$\mathbf{r}(t) = \int \mathrm{v}(t)\,\mathrm{d}t = \left[\int (t+2)\,\mathrm{d}t\right]\mathrm{i} + \left(\int t^2\,\mathrm{d}t\right)\mathrm{j} + \left(\int e^{-t/3}\,\mathrm{d}t\right)\mathbf{k}$$

$$\mathbf{r}(t) = \left(\frac{1}{2}t^2 + 2t + c_1\right)\mathbf{i} + \left(\frac{1}{3}t^3 + c_2\right)\mathbf{j} + \left(-3e^{-t/3} + c_3\right)\mathbf{k}$$

Using the initial condition gives

$$\mathbf{r}(0) = 4\,\mathrm{i} + 0\,\mathrm{j} - 2\,\mathrm{k} = c_1\,\mathbf{i} + c_2\,\mathbf{j} + (-3 + c_3)\,\mathbf{k} \implies c_1 = 4, c_2 = 0, c_3 = 1$$

so that

$$\mathbf{r}(t) = \left(\frac{1}{2}t^2 + 2t + 4\right)\mathbf{i} + \left(\frac{1}{3}t^3\right)\mathbf{j} + \left(-3e^{-t/3} + 1\right)\mathbf{k}$$

(c) i.

$$\mathbf{r}'(t) = \mathbf{i} + (2t - 2)\,\mathbf{j}$$

$$\mathbf{r}''(t) = 2\,\mathbf{j}$$

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\|\,[\mathbf{i} + (2t-2)\,\mathbf{j}] \times 2\,\mathbf{j}\|}{\sqrt{1 + (2t-2)^2}} = \frac{2}{(4t^2 - 8t + 5)^{3/2}}$$

$$\kappa(1) = 2$$

ii. To determine if the tangential component of the acceleration is ever zero, we have

$$a_T(t) = \frac{\mathrm{d}\|\mathbf{v}\|}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(4t^2 - 8t + 5)^{1/2} = \frac{4(t-1)}{\sqrt{4t^2 - 8t + 5}} \implies a_T(t) = 0 \text{ if } t = 1$$

Thus, when $t = 1$ the acceleration will consist of only a normal component. We have

$$a_N(t) = \kappa\|\mathbf{v}\|^2 = \frac{2}{(4t^2 - 8t + 5)^{3/2}}\left(\sqrt{4t^2 - 8t + 5}\right)^2 = \frac{2}{\sqrt{4t^2 - 8t + 5}} \implies a_N(1) = 2$$

The point on the curve where this occurs is $\mathbf{r}(1) = 2\,\mathbf{i} - \mathbf{j}$ or $(2, -1, 0)$.