NOTE: Any integrals that need to be evaluated will require integration techniques no more complicated than u-substitution or the use of trigonometric identities.

1. [40 pts] In your bluebook, write the word **TRUE** if the statement is always true or write the word **FALSE** if the statement is false. No justification needed and no partial credit given.

   (a) The path $\mathbf{r}(t) = \sin(t^2)\,\mathbf{i} - \cos(t^2)\,\mathbf{j}$ with $0 \le t \le \sqrt{2\pi}$ is an arc length parameterization of the unit circle.

   (b) The function $g(x, y) = x^2 + 4xy - 2y^2 + 6x - 12y$ has a local maximum or minimum value at $(x, y) = (1, -2)$.

   (c) A direction vector for the line normal to the level surface of $F(x, y, z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9}$ at the point $(-2, 1, -3)$ is $\left\langle -1, 2, -\frac{2}{3}\right\rangle$.

   (d) For a path $\mathbf{r}(t)$ lying entirely in the $yz-$plane, the unit binormal $\mathbf{B} = \mathbf{j}$.

   (e) Let $h(x, y, z)$ be a differentiable function, and $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ with $x(t), y(t)$ and $z(t)$ differentiable functions of t. Then $\nabla h \cdot \mathbf{r}'(t) = \dfrac{\mathrm{d}h}{\mathrm{ds}}\|\mathbf{r}'(t)\|$.

   (f) A particle moving at constant speed along a path with zero curvature will experience no normal acceleration.

   (g) The vector $\langle 3, -1, 2\rangle$ is parallel to the plane $6x - 2y + 4z = 1$.

   (h) $f(x, y) = 1 + x + xy$ could be the linearization or first order Taylor polynomial for some function.

SOLUTION:

   (a) **FALSE** $\|\mathbf{r}'(t)\| = 2t \ne 1$

   (b) **FALSE** $D(x, y) = g_{xx}g_{yy} - g_{xy}^2 = -8 < 0$ so any critical points will be saddles.

   (c) **TRUE** $\nabla F(-2, 1, -3) = \left\langle -1, 2, -\frac{2}{3}\right\rangle$

   (d) **FALSE** $\mathbf{B} = \pm\mathbf{i}$

   (e) **TRUE** The equation is equivalent to $\dfrac{\mathrm{d}h}{\mathrm{d}t} = \dfrac{\mathrm{d}h}{\mathrm{d}s}\dfrac{\mathrm{d}s}{\mathrm{d}t}$

   (f) **TRUE** Normal acceleration is proportional to the curvature.

   (g) **FALSE** $\langle 3, -1, 2\rangle = \frac{1}{2}\langle 6, -2, 4\rangle$ so it is a scalar multiple of the plane's normal vector and consequently normal to the plane.

   (h) **FALSE** The xy term is not linear.

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2. [30 pts] The following problems are unrelated.

   (a) [5 pts] Let $\mathbf{V} = (x + 2y + az)\,\mathbf{i} + (bx - 3y - z)\,\mathbf{j} + (4x + cy + 2z)\,\mathbf{k}$. Find constants a, b and c so that $\displaystyle\int_{\mathcal{C}} \mathbf{V} \cdot \mathrm{d}\mathbf{r}$ is path independent.

   (b) [25 pts] Find the work done by the conservative force field $\mathbf{F} = \left(2xy + z^3\right)\mathbf{i} + x^2\,\mathbf{j} + 3xz^2\,\mathbf{k}$ in moving an object along the path $\mathbf{r}(t) = \frac{4}{\pi}\left\langle \sin^{-1} t, \tan^{-1} t, \cos^{-1} t\right\rangle,\ -1 \le t \le 1$. Simplify your answer completely (*i.e.*, eliminate all trig functions)

SOLUTION:

   (a) In order for the integral to be path independent, the curl of $\mathbf{V}$ must vanish on a simply connected domain. Since $\mathbf{V}$ exists on $\mathbb{R}^3$, which is simply connected, we need to force the curl to vanish throughout $\mathbb{R}^3$.

$$\nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x + 2y + az & bx - 3y - z & 4x + cy + 2z \end{vmatrix} = (c + 1)\,\mathbf{i} + (a - 4)\,\mathbf{j} + (b - 2)\,\mathbf{k}$$

   The curl will vanish if $a = 4, b = 2, c = -1$.

   (b) We need to compute $\displaystyle\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$, which looks impossible given the path. However, since the vector field is conservative we can use the Fundamental Theorem for Line Integrals to get the answer. We find the potential for the vector field:

$$\frac{\partial f}{\partial x} = 2xy + z^3 \implies f(x, y, z) = \int \left(2xy + z^3\right)\,\mathrm{d}x = x^2y + xz^3 + g(y, z)$$

$$\frac{\partial f}{\partial y} = x^2 + g_y(y, z) = x^2 \implies g_y(y, z) = 0 \implies g(y, z) = h(z) \implies f(x, y, z) = x^2y + xz^3 + h(z)$$

$$\frac{\partial f}{\partial z} = 3xz^2 + h'(z) = 3xz^2 \implies h'(z) = 0 \implies h(z) = c \quad \text{which we set to } 0$$

The potential function is then $f(x,y,z) = x^2y + xz^3$. The endpoints of the path are

$$\mathbf{r}(-1) = \frac{4}{\pi}\left\langle \sin^{-1}(-1), \tan^{-1}(-1), \cos^{-1}(-1)\right\rangle = \langle -2,-1,4\rangle$$
$$\mathbf{r}(1) = \frac{4}{\pi}\left\langle \sin^{-1}(1), \tan^{-1}(1), \cos^{-1}(1)\right\rangle = \langle 2,1,0\rangle$$

We then have

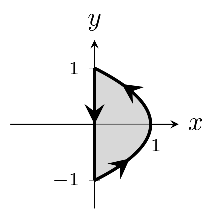
$$\int_{\mathcal{C}} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_{\langle -2,-1,4\rangle}^{\langle 2,1,0\rangle} \nabla f \cdot \mathrm{d}\mathbf{r} = f(2,1,0) - f(-2,-1,4)$$
$$= 2^2(1) + 2\left(0^3\right) - \left[(-2)^2(-1) + (-2)(4)^3\right] = 136$$

3. [30 pts] Consider the vector field $\mathbf{V} = x\,\mathbf{i} + y^2\,\mathbf{j}$. Let $\mathcal{D}$ be the finite region bounded on the left by the $y-$axis and on the right by $x = 1 - y^2$ with boundary curve $\mathcal{C}$ oriented counterclockwise.

(a) [15 pts] Compute the circulation (flow) of $\mathbf{V}$ along $\mathcal{C}$ directly, that is, without the use of any theorems.

(b) [15 pts] Compute the flux of $\mathbf{V}$ through $\mathcal{C}$ using Green's theorem.

SOLUTION:



(a) The boundary curve is piecewise smooth, consisting of two smooth pieces, $\mathcal{C}_1$ and $\mathcal{C}_2$, with parameterizations

$$\mathbf{r}_1(t) = \langle 0, 1-2t\rangle,\ 0 \le t \le 1 \text{ and } \mathbf{r}_2(t) = \langle 1-t^2, t\rangle,\ -1 \le t \le 1$$

respectively. We then have

$$\text{Circulation} = \int_{\mathcal{C}} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_{\mathcal{C}_1} \mathbf{F}\cdot \mathrm{d}\mathbf{r} + \int_{\mathcal{C}_2} \mathbf{F}\cdot \mathrm{d}\mathbf{r}$$

For $\mathcal{C}_1$:

$$\mathbf{F}(\mathbf{r}_1(t))\cdot \mathbf{r}'(t) = \langle 0, (1-2t)^2\rangle \cdot \langle 0,-2\rangle = -2 + 8t - 8t^2$$
$$\Longrightarrow \int_{\mathcal{C}_1} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_0^1 \left(-2+8t-8t^2\right)\mathrm{d}t = \left(-2t + 4t^2 - \frac{8}{3}t^3\right)\Bigg|_0^1 = -\frac{2}{3}$$

For $\mathcal{C}_2$:

$$\mathbf{F}(\mathbf{r}_2(t))\cdot \mathbf{r}'(t) = \langle 1-t^2, t^2\rangle \cdot \langle -2t, 1\rangle = -2t + 2t^3 + t^2$$
$$\Longrightarrow \int_{\mathcal{C}_2} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_{-1}^1 \left(-2t + 2t^3 + t^2\right)\mathrm{d}t = \frac{2}{3}t^3\Big|_0^1 = \frac{2}{3}$$

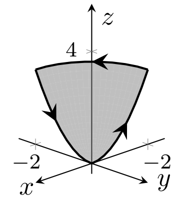
Thus

$$\text{Circulation} = \int_{\mathcal{C}} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_{\mathcal{C}_1} \mathbf{F}\cdot \mathrm{d}\mathbf{r} + \int_{\mathcal{C}_2} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = -\frac{2}{3} + \frac{2}{3} = 0$$

(b) We have $\nabla \cdot \mathbf{F} = 1 + 2y$ and, using Green's Theorem,

$$\text{Flux} = \int_{\mathcal{C}} \mathbf{F}\cdot \mathbf{n}\,\mathrm{d}s = \iint_{\mathcal{D}} \nabla \cdot \mathbf{F}\,\mathrm{d}A = \int_{-1}^1 \int_0^{1-y^2} (1+2y)\,\mathrm{d}x\,\mathrm{d}y$$
$$= \int_{-1}^1 \left(1 - y^2 + 2y - 2y^3\right)\mathrm{d}y = 2\left(y - \frac{y^3}{3}\right)\Bigg|_0^1 = \frac{4}{3}$$

4. [15 pts] Let $\mathcal{C}$ be the curve bounding the portion of the paraboloid $2z = x^2 + y^2$ below $z = 2$ and lying above the third quadrant, oriented as shown in the following figure. Compute $\int_{\mathcal{C}} yz\,\mathrm{d}x - xz\,\mathrm{d}y + z^3\,\mathrm{d}z$ using Stokes' Theorem.



SOLUTION:

Note that Stokes' Theorem gives:

$$\int_{\mathcal{C}} yz\,\mathrm{d}x - xz\,\mathrm{d}y + z^3\,\mathrm{d}z = \int_{\mathcal{C}} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \iint_{\mathcal{S}} \nabla\times\mathbf{F}\cdot \mathrm{d}\mathbf{S}$$

where $\mathbf{F} = \langle yz, -xz, z^3\rangle$ and $\mathcal{S}$ is the paraboloid.

$$g(x,y,z) = 2z - x^2 - y^2 \implies \nabla g = \langle -2x, -2y, 2\rangle \quad \text{use } +\nabla g \text{ for consistent orientation}$$

and by projecting onto the $xy-$plane $\mathbf{p} = \mathbf{k}$ so that $|\nabla g\cdot\mathbf{p}| = 2$ and the region of integration is the third quadrant portion of the disk of radius 2 centered at the origin.

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & -xz & z^3 \end{vmatrix} = x\,\mathbf{i} + y\,\mathbf{j} - 2z\,\mathbf{k}$$

$$\nabla\times\mathbf{F}\cdot\frac{+\nabla g}{|\nabla g\cdot\mathbf{p}|} = -x^2 - y^2 - 2z = -2(x^2+y^2)$$

where we have used the surface to eliminate z since integration will take place in the $xy-$plane. We then have

$$\int_{\mathcal{C}} yz\,\mathrm{d}x - xz\,\mathrm{d}y + z^3\,\mathrm{d}z = \int_{-2}^{0}\int_{-\sqrt{4-x^2}}^{0} -2\left(x^2+y^2\right)\mathrm{d}y\,\mathrm{d}x$$

$$= -2\int_{\pi}^{3\pi/2}\int_0^2 r^3\,\mathrm{d}r\,\mathrm{d}\theta = -4\pi$$

5. [35 pts] You need to compute the downward flux of $\mathbf{F} = \frac{x}{\sqrt{x^2+y^2+z^2}}\,\mathbf{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\,\mathbf{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\,\mathbf{k}$ through that portion of the sphere of radius 2 centered at the origin lying below the plane $z = -1$ (not a closed surface). To accomplish this feat, let $\mathcal{E}$ be the solid region between the aforementioned sphere and plane. Its closed boundary, $\mathcal{S}$, consists of the union of a disk, $\mathcal{S}_{\text{disk}}$, and the surface through which we seek the flux, $\mathcal{S}_{\text{sphere}}$, that is, $\mathcal{S} = \mathcal{S}_{\text{disk}} \cup \mathcal{S}_{\text{sphere}}$.

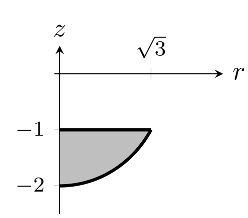
(a) [15 pts] Find the upward flux of $\mathbf{F}$ through $\mathcal{S}_{\text{disk}}$.

(b) [5 pts] Show that the divergence of $\mathbf{F}$ is $\frac{2}{\sqrt{x^2+y^2+z^2}}$.

(c) [15 pts] Use Gauss' Divergence Theorem to find $\iint_{\mathcal{S}_{\text{sphere}}} \mathbf{F}\cdot\mathrm{d}\mathbf{S}$. (No points for calculating the flux directly.)

SOLUTION:

A sketch in the $rz-$plane will be beneficial.



(a) The surface $\mathcal{S}_{\text{disk}}$ is a portion of the level surface $g(x, y, z) = z$ where $z = -1$, giving $\nabla g = \mathbf{k}$. Projecting onto the $xy-$plane makes $\mathbf{p} = \mathbf{k}$ and $|\nabla g \cdot \mathbf{p}| = 1$. The region of integration is determined with $x^2 + y^2 + (-1)^2 = 4$ and is the disk $x^2 + y^2 \leq 3$. To find the upward flux we choose $+\nabla g$ giving (eliminating z using the surface)

$$\mathbf{F} \cdot \frac{\nabla g}{|\nabla g \cdot \mathbf{p}|} = \frac{z}{\sqrt{x^2 + y^2 + z^2}} = \frac{-1}{\sqrt{x^2 + y^2 + 1}}$$

Thus

$$\iint_{\mathcal{S}_{\text{disk}}} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iint_{x^2+y^2\leq 3} \frac{-1}{\sqrt{x^2 + y^2 + 1}} \,\mathrm{d}A = -\int_0^{2\pi} \int_0^{\sqrt{3}} \frac{r}{\sqrt{r^2 + 1}} \,\mathrm{d}r\,\mathrm{d}\theta = \left(-\int_0^{2\pi} \mathrm{d}\theta\right) \left(\int_1^4 \frac{1}{2} u^{-1/2} \,\mathrm{d}u\right) = -2\pi$$

(b)

$$\begin{aligned}
\nabla \cdot \mathbf{F} &= \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2 + y^2 + z^2}}\right) + \frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2 + y^2 + z^2}}\right) + \frac{\partial}{\partial z}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right) \\
&= \frac{\sqrt{x^2 + y^2 + z^2} - x\left(\frac{2x}{2\sqrt{x^2+y^2+z^2}}\right)}{x^2 + y^2 + z^2} + \frac{\sqrt{x^2 + y^2 + z^2} - y\left(\frac{2y}{2\sqrt{x^2+y^2+z^2}}\right)}{x^2 + y^2 + z^2} + \frac{\sqrt{x^2 + y^2 + z^2} - z\left(\frac{2z}{2\sqrt{x^2+y^2+z^2}}\right)}{x^2 + y^2 + z^2} \\
&= \frac{y^2 + z^2}{(x^2 + y^2 + z^2)^{3/2}} + \frac{x^2 + z^2}{(x^2 + y^2 + z^2)^{3/2}} + \frac{x^2 + y^2}{(x^2 + y^2 + z^2)^{3/2}} \\
&= \frac{2x^2 + 2y^2 + 2z^2}{(x^2 + y^2 + z^2)\sqrt{x^2 + y^2 + z^2}} = \frac{2}{\sqrt{x^2 + y^2 + z^2}}
\end{aligned}$$

(c) Gauss' Divergence Theorem will be used in the form

$$\iint_{\mathcal{S}_{\text{sphere}}} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iiint_{\mathcal{E}} \nabla \cdot \mathrm{F} \,\mathrm{d}V - \iint_{\mathcal{S}_{\text{disk}}} \mathbf{F} \cdot \mathrm{d}\mathbf{S}$$

Now, transforming to spherical coordinates we have

$$\begin{aligned}
\iiint_{\mathcal{E}} \nabla \cdot \mathrm{F} \,\mathrm{d}V &= \iiint_{\mathcal{E}} \frac{2}{\sqrt{x^2 + y^2 + z^2}} \,\mathrm{d}V = \int_0^{2\pi} \int_{2\pi/3}^{\pi} \int_{-\sec\phi}^{2} \frac{2}{\rho} \rho^2 \sin\phi \,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta \\
&= \left(\int_0^{2\pi} \mathrm{d}\theta\right) \int_{2\pi/3}^{\pi} \int_{-\sec\phi}^{2} 2\rho \sin\phi \,\mathrm{d}\rho\,\mathrm{d}\phi = 2\pi \int_{2\pi/3}^{\pi} \rho^2 \Big|_{-\sec\phi}^{2} \sin\phi \,\mathrm{d}\phi \\
&= 2\pi \int_{2\pi/3}^{\pi} \left(4\sin\phi - \sec^2\phi \sin\phi\right) \mathrm{d}\phi = 2\pi \left(4\cos\phi\Big|_{\pi}^{2\pi/3} - \int_{2\pi/3}^{\pi} \sec\phi \tan\phi \,\mathrm{d}\phi\right) \\
&= 2\pi \left(2 - \sec\phi\Big|_{2\pi/3}^{\pi}\right) = 2\pi
\end{aligned}$$

Finally then,

$$\iint_{\mathcal{S}_{\text{sphere}}} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = 2\pi - (-2\pi) = 4\pi$$

Alternatively, cylindrical coordinates give

$$\int_0^{2\pi} \int_0^{\sqrt{3}} \int_{-\sqrt{4-r^2}}^{-1} \frac{2}{\sqrt{r^2 + z^2}} r \,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta \qquad \text{difficult integral}$$

or

$$\begin{aligned}
&\int_0^{2\pi} \int_{-2}^{-1} \int_0^{\sqrt{4-z^2}} \frac{2}{\sqrt{r^2 + z^2}} r \,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}\theta \overset{u=r^2+z^2}{=} \int_0^{2\pi} \int_{-2}^{-1} \int_{z^2}^{4} u^{-1/2} \,\mathrm{d}u\,\mathrm{d}z\,\mathrm{d}\theta = 2\pi \int_{-2}^{-1} 2\sqrt{u}\Big|_{z^2}^{4} \mathrm{d}z \\
&= 4\pi \int_{-2}^{-1} (2 - |z|) \,\mathrm{d}z = 4\pi \int_{-2}^{-1} (2 + z) \,\mathrm{d}z = 4\pi \left(2z + \tfrac{z^2}{2}\right)\Big|_{-2}^{-1} = 4\pi \left[-2 + \tfrac{1}{2} - (-4 + 2)\right] = 2\pi
\end{aligned}$$

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