NOTE: Any integrals that need to be evaluated will require integration techniques no more complicated than \( u \)-substitution or the use of trigonometric identities.

1. [40 pts] In your bluebook, write the word TRUE if the statement is always true or write the word FALSE if the statement is false. No justification needed and no partial credit given.

   (a) The path \( r(t) = \sin(t^2) \mathbf{i} - \cos(t^2) \mathbf{j} \) with \( 0 \leq t \leq \sqrt{2\pi} \) is an arc length parameterization of the unit circle.

   (b) The function \( g(x, y) = x^2 + 4xy - 2y^2 + 6x - 12y \) has a local maximum or minimum value at \( (x, y) = (1, -2) \).

   (c) A direction vector for the line normal to the level surface of \( F(x, y, z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9} \) at the point \( (-2, 1, -3) \) is \( \langle -1, 2, -\frac{2}{3} \rangle \).

   (d) For a path \( r(t) \) lying entirely in the \( yz \)-plane, the unit binormal \( \mathbf{B} = \mathbf{j} \).

   (e) Let \( h(x, y, z) \) be a differentiable function, and \( r(t) = \langle x(t), y(t), z(t) \rangle \) with \( x(t), y(t) \) and \( z(t) \) differentiable functions of \( t \). Then \( \nabla h \cdot r'(t) = \frac{dh}{dt} \| r'(t) \| \).

   (f) A particle moving at constant speed along a path with zero curvature will experience no normal acceleration.

   (g) The vector \( (3, -1, 2) \) is parallel to the plane \( 6x - 2y + 4z = 1 \).

   (h) \( f(x, y) = 1 + x + xy \) could be the linearization or first order Taylor polynomial for some function.

2. [30 pts] The following problems are unrelated.

   (a) [5 pts] Let \( \mathbf{V} = (x + 2y + az) \mathbf{i} + (bx - 3y - z) \mathbf{j} + (4x + cy + 2z) \mathbf{k} \). Find constants \( a, b \) and \( c \) so that \( \int_{C} \mathbf{V} \cdot d\mathbf{r} \) is path independent.

   (b) [25 pts] Find the work done by the conservative force field \( \mathbf{F} = (2xy + z^3) \mathbf{i} + x^2 \mathbf{j} + 3xz^2 \mathbf{k} \) in moving an object along the path \( r(t) = \frac{1}{t} \langle \sin^{-1} t, \tan^{-1} t, \cos^{-1} t \rangle \), \( -1 \leq t \leq 1 \). Simplify your answer completely (i.e., eliminate all trig functions)

3. [30 pts] Consider the vector field \( \mathbf{V} = x \mathbf{i} + y^2 \mathbf{j} \). Let \( D \) be the finite region bounded on the left by the \( y \)-axis and on the right by \( x = 1 - y^2 \) with boundary curve \( \mathcal{C} \) oriented counterclockwise.

   (a) [15 pts] Compute the circulation (flow) of \( \mathbf{V} \) along \( \mathcal{C} \) directly, that is, without the use of any theorems.

   (b) [15 pts] Compute the flux of \( \mathbf{V} \) through \( \mathcal{C} \) using Green’s theorem.

4. [15 pts] Let \( \mathcal{C} \) be the curve bounding the portion of the paraboloid \( 2z = x^2 + y^2 \) below \( z = 2 \) and lying above the third quadrant, oriented as shown in the following figure. Compute \( \int_{\mathcal{C}} yz \, dx - xz \, dy + z^3 \, dz \) using Stokes’ Theorem.

   ![Diagram of a paraboloid with a curve bounding the portion below z = 2 and above the third quadrant]

5. [35 pts] You need to compute the downward flux of \( \mathbf{F} = \frac{x}{\sqrt{x^2+y^2+z^2}} \mathbf{i} + \frac{y}{\sqrt{x^2+y^2+z^2}} \mathbf{j} + \frac{z}{\sqrt{x^2+y^2+z^2}} \mathbf{k} \) through that portion of the sphere of radius 2 centered at the origin lying below the plane \( z = -1 \) (not a closed surface). To accomplish this feat, let \( \mathcal{E} \) be the solid region between the aforementioned sphere and plane. Its closed boundary, \( \mathcal{S} \), consists of the union of a disk, \( \mathcal{S}_{\text{disk}} \), and the surface through which we seek the flux, \( \mathcal{S}_{\text{sphere}} \), that is, \( \mathcal{S} = \mathcal{S}_{\text{disk}} \cup \mathcal{S}_{\text{sphere}} \).

   (a) [15 pts] Find the upward flux of \( \mathbf{F} \) through \( \mathcal{S}_{\text{disk}} \).

   (b) [5 pts] Show that the divergence of \( \mathbf{F} \) is \( \frac{2}{\sqrt{x^2+y^2+z^2}} \).

   (c) [15 pts] Use Gauss’ Divergence Theorem to find \( \int_{\mathcal{S}_{\text{sphere}}} \mathbf{F} \cdot d\mathbf{S} \). (No points for calculating the flux directly.)

FORMULAS ON BACK
PROJECTION; DISTANCE FROM POINT S TO LINE PARALLEL TO \( \mathbf{v} \) CONTAINING POINT P; DISTANCE FROM POINT S TO PLANE WITH NORMAL \( \mathbf{n} \) CONTAINING POINT P

\[
\text{proj}_\mathbf{b} \mathbf{a} = \left( \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}} \right) \mathbf{a}
\]

\[
d = \left| \frac{\mathbf{F}_S \times \mathbf{v}}{\| \mathbf{v} \|} \right|
\]

ARC LENGTH, FRENET FORMULAS, AND TANGENTIAL AND NORMAL ACCELERATION COMPONENTS

\[
ds = \| \mathbf{v} \| \, dt \quad \mathbf{T} = \frac{d \mathbf{r}}{ds} \quad \mathbf{N} = \frac{d \mathbf{T}/ds}{\| d \mathbf{T}/ds \|} \quad \mathbf{B} = \mathbf{T} \times \mathbf{N}
\]

\[
\frac{dT}{ds} = \kappa \mathbf{N} \quad \frac{dB}{ds} = -\tau \mathbf{N} \quad \kappa = \frac{\| d\mathbf{T}/ds \|}{\| \mathbf{v} \|} = \frac{\| \mathbf{v} \times \mathbf{a} \|}{\| \mathbf{v} \| \| \mathbf{a} \|} = \frac{\| f''(x) \|}{\left( 1 + [f'(x)]^2 \right)^{3/2}} = \frac{\| \hat{x}\mathbf{y} - \hat{y}\mathbf{x} \|}{(x^2 + y^2)^{3/2}} \quad \tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N}
\]

DIRECTIONAL DERIVATIVE, LAGRANGE MULTIPLIERS

\[
Da f = \frac{df}{ds} = \nabla f \cdot \mathbf{u} \quad \nabla f = \lambda \nabla g, \quad g = 0
\]

SECOND DERIVATIVES TEST: Suppose \( f(x, y) \) and its first and second partial derivatives are continuous in a disk centered at \((a, b)\) and \( f_x(a, b) = f_y(a, b) = 0 \). Let \( D = f_{xx}f_{yy} - f_{xy}^2 \).
- If \( D > 0 \) and \( f_{xx} < 0 \) at \((a, b)\), then \( f \) has a local maximum at \((a, b)\).
- If \( D > 0 \) and \( f_{xx} > 0 \) at \((a, b)\), then \( f \) has a local minimum at \((a, b)\).
- If \( D < 0 \) at \((a, b)\), then \( f \) has a saddle point at \((a, b)\).
- If \( D = 0 \) at \((a, b)\), then the test is inconclusive.

TAYLOR’S FORMULA [at the point \((x_0, y_0)\)]

\[
f(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + \frac{1}{2!} \left[ f_{xx}(x_0, y_0)(x - x_0)^2 + 2f_{xy}(x_0, y_0)(x - x_0)(y - y_0) + f_{yy}(x_0, y_0)(y - y_0)^2 \right] + \frac{1}{3!} \left[ f_{xxx}(x_0, y_0)(x - x_0)^3 + 3f_{xxy}(x_0, y_0)(x - x_0)^2(y - y_0) + 3f_{xyy}(x_0, y_0)(x - x_0)(y - y_0)^2 + f_{yyy}(x_0, y_0)(y - y_0)^3 \right] + \cdots
\]

ERROR IN LINEAR APPROXIMATION

\[
|E(x, y)| \leq \frac{1}{2!} M \left( \| x - x_0 \| + \| y - y_0 \|^2 \right) \quad \text{where} \quad \max \{ |f_{xx}|, |f_{xy}|, |f_{yy}| \} \leq M
\]

ERROR IN QUADRATIC APPROXIMATION

\[
|E(x, y)| \leq \frac{1}{3!} M \left( \| x - x_0 \| + \| y - y_0 \|^3 \right) \quad \text{where} \quad \max \{ |f_{xx}|, |f_{xy}|, |f_{yy}|, |f_{yyyy}| \} \leq M
\]

CHANGE OF VARIABLES/SUBSTITUTIONS IN MULTIPLE INTEGRALS

\[
\iiint_R f(x, y) \, dA = \iint_S f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \, du \, dv \quad \text{where} \quad J(u, v) = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| = \left| \frac{\partial u}{\partial \rho} \frac{\partial \rho}{\partial \phi} - \frac{\partial u}{\partial \phi} \frac{\partial \rho}{\partial \theta} \right| = \frac{\partial x}{\partial u} \frac{\partial y}{\partial u} - \frac{\partial x}{\partial u} \frac{\partial y}{\partial v}
\]

POLAR COORDINATES

\[
x = r \cos \theta \quad y = r \sin \theta \quad r^2 = x^2 + y^2 \quad dA = dx \, dy = r \, dr \, d\theta
\]

Coordinate Conversions

<table>
<thead>
<tr>
<th>Cylindrical to Rectangular</th>
<th>Spherical to Rectangular</th>
<th>Spherical to Cylindrical</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x = r \cos \theta )</td>
<td>( x = \rho \sin \phi \cos \theta )</td>
<td>( r = \rho \sin \phi )</td>
</tr>
<tr>
<td>( y = r \sin \theta )</td>
<td>( y = \rho \sin \phi \sin \theta )</td>
<td>( y = \rho \cos \phi )</td>
</tr>
<tr>
<td>( z = z )</td>
<td>( z = \rho \cos \phi )</td>
<td>( z = \rho \cos \phi )</td>
</tr>
</tbody>
</table>

\[
dV = dx \, dy \, dz = r \, dr \, d\theta \, dz = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta
\]

MASS, MOMENTS, AND CENTER OF MASS

\[
M = \iiint_R \delta \, dA \quad \text{Moments} \quad M_x = \iiint_R y \, \delta \, dA \quad M_y = \iiint_R x \, \delta \, dA \quad \text{Center of mass} \quad \mathbf{x} = \frac{M_y}{M} \quad \mathbf{y} = \frac{M_x}{M}
\]

FLOW AND FLUX

\[
\text{Flow} = \int_C \mathbf{F} \cdot \mathbf{T} \, ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P \, dx + Q \, dy \quad \text{Flux} = \int_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_C P \, dy - Q \, dx \quad \mathbf{n} = \mathbf{T} \times \mathbf{k} \quad \mathbf{F}(x, y) = P(x, y) \mathbf{i} + Q(x, y) \mathbf{j}
\]

GREEN’S THEOREM

\[
C \quad \text{is the boundary curve enclosing the region} \ D, \ \text{traversed counterclockwise.}
\]

\[
\text{FLOW/TANGENTIAL FORM} \quad \iiint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA = \oint_C P \, dx + Q \, dy \quad \text{FLUX/NORMAL FORM} \quad \iiint_D \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \, dA = \int_C P \, dy - Q \, dx
\]

SURFACE AREA OF LEVEL SURFACE \( g(x, y, z) = c \)

\[
S = \iint_S dS = \iint_R \frac{\| \nabla g \|}{\| \nabla g \cdot \mathbf{p} \|} \, dA
\]

STOKES' THEOREM

\[
\iint_S (\nabla \times \mathbf{F}) \cdot dS = \oint_C \mathbf{F} \cdot d\mathbf{r} \quad C \quad \text{is the boundary curve of the surface} \ S
\]

GAUSS’ DIVERGENCE THEOREM

\[
\iiint_E \mathbf{\nabla} \cdot \mathbf{F} \, dV = \iint_S \mathbf{F} \cdot dS \quad S \quad \text{is the boundary surface of the solid} \ E
\]

FUN TRIGONOMETRY FACTS

\[
\sin^2 x = \frac{1 - \cos 2x}{2} \quad \cos^2 x = \frac{1 + \cos 2x}{2}
\]