Write on the front of your bluebook a grading key (using the printed lines), your name, your lecture section number and instructor. This exam is worth 150 points and has 5 questions.

- Submit this exam sheet with your bluebook. However, nothing on this exam sheet will be graded. Make sure all of your work is in your bluebook.
- **Show all work and simplify your answers!** Answers with no justification will receive no points unless otherwise noted. **Please begin each problem on a new page.**
- You are taking this exam in a proctored and honor code enforced environment. Thus, no notes/papers, calculators, cell phones, or other electronic devices are permitted.

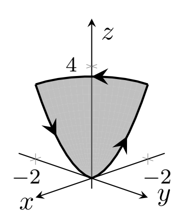
NOTE: Any integrals that need to be evaluated will require integration techniques no more complicated than u-substitution or the use of trigonometric identities.

1. [40 pts] In your bluebook, write the word **TRUE** if the statement is always true or write the word **FALSE** if the statement is false. No justification needed and no partial credit given.
   (a) The path $\mathbf{r}(t) = \sin(t^2)\,\mathbf{i} - \cos(t^2)\,\mathbf{j}$ with $0 \le t \le \sqrt{2\pi}$ is an arc length parameterization of the unit circle.
   (b) The function $g(x, y) = x^2 + 4xy - 2y^2 + 6x - 12y$ has a local maximum or minimum value at $(x, y) = (1, -2)$.
   (c) A direction vector for the line normal to the level surface of $F(x, y, z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9}$ at the point $(-2, 1, -3)$ is $\left\langle -1, 2, -\frac{2}{3} \right\rangle$.
   (d) For a path $\mathbf{r}(t)$ lying entirely in the $yz-$plane, the unit binormal $\mathbf{B} = \mathbf{j}$.
   (e) Let $h(x, y, z)$ be a differentiable function, and $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ with $x(t), y(t)$ and $z(t)$ differentiable functions of t. Then $\nabla h \cdot \mathbf{r}'(t) = \dfrac{\mathrm{d}h}{\mathrm{ds}} \|\mathbf{r}'(t)\|$.
   (f) A particle moving at constant speed along a path with zero curvature will experience no normal acceleration.
   (g) The vector $\langle 3, -1, 2 \rangle$ is parallel to the plane $6x - 2y + 4z = 1$.
   (h) $f(x, y) = 1 + x + xy$ could be the linearization or first order Taylor polynomial for some function.

2. [30 pts] The following problems are unrelated.
   (a) [5 pts] Let $\mathbf{V} = (x + 2y + az)\,\mathbf{i} + (bx - 3y - z)\,\mathbf{j} + (4x + cy + 2z)\,\mathbf{k}$. Find constants a, b and c so that $\int_{\mathcal{C}} \mathbf{V} \cdot \mathrm{d}\mathbf{r}$ is path independent.
   (b) [25 pts] Find the work done by the conservative force field $\mathbf{F} = \left(2xy + z^3\right)\mathbf{i} + x^2\,\mathbf{j} + 3xz^2\,\mathbf{k}$ in moving an object along the path $\mathbf{r}(t) = \frac{4}{\pi}\left\langle \sin^{-1} t, \tan^{-1} t, \cos^{-1} t \right\rangle,\ -1 \le t \le 1$. Simplify your answer completely (*i.e.*, eliminate all trig functions)

3. [30 pts] Consider the vector field $\mathbf{V} = x\,\mathbf{i} + y^2\,\mathbf{j}$. Let $\mathcal{D}$ be the finite region bounded on the left by the $y-$axis and on the right by $x = 1 - y^2$ with boundary curve $\mathcal{C}$ oriented counterclockwise.
   (a) [15 pts] Compute the circulation (flow) of $\mathbf{V}$ along $\mathcal{C}$ directly, that is, without the use of any theorems.
   (b) [15 pts] Compute the flux of $\mathbf{V}$ through $\mathcal{C}$ using Green's theorem.

4. [15 pts] Let $\mathcal{C}$ be the curve bounding the portion of the paraboloid $2z = x^2 + y^2$ below $z = 2$ and lying above the third quadrant, oriented as shown in the following figure. Compute $\int_{\mathcal{C}} yz\,\mathrm{d}x - xz\,\mathrm{d}y + z^3\,\mathrm{d}z$ using Stokes' Theorem.



5. [35 pts] You need to compute the downward flux of $\mathbf{F} = \frac{x}{\sqrt{x^2+y^2+z^2}}\,\mathbf{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\,\mathbf{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\,\mathbf{k}$ through that portion of the sphere of radius 2 centered at the origin lying below the plane $z = -1$ (not a closed surface). To accomplish this feat, let $\mathcal{E}$ be the solid region between the aforementioned sphere and plane. Its closed boundary, $\mathcal{S}$, consists of the union of a disk, $\mathcal{S}_{\text{disk}}$, and the surface through which we seek the flux, $\mathcal{S}_{\text{sphere}}$, that is, $\mathcal{S} = \mathcal{S}_{\text{disk}} \cup \mathcal{S}_{\text{sphere}}$.
   (a) [15 pts] Find the upward flux of $\mathbf{F}$ through $\mathcal{S}_{\text{disk}}$.
   (b) [5 pts] Show that the divergence of $\mathbf{F}$ is $\frac{2}{\sqrt{x^2+y^2+z^2}}$.
   (c) [15 pts] Use Gauss' Divergence Theorem to find $\iint_{\mathcal{S}_{\text{sphere}}} \mathbf{F} \cdot \mathrm{d}\mathbf{S}$. (No points for calculating the flux directly.)

FORMULAS ON BACK

PROJECTION; DISTANCE FROM POINT S TO LINE PARALLEL TO $\mathbf{v}$ CONTAINING POINT P; DISTANCE FROM POINT S TO PLANE WITH NORMAL $\mathbf{n}$ CONTAINING POINT P

$$\text{proj}_{\mathbf{a}}\mathbf{b} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a} \qquad d = \frac{\left\|\overrightarrow{PS}\times\mathbf{v}\right\|}{\|\mathbf{v}\|} \qquad d = \left|\overrightarrow{PS}\cdot\frac{\mathbf{n}}{\|\mathbf{n}\|}\right|$$

ARC LENGTH, FRENET FORMULAS, AND TANGENTIAL AND NORMAL ACCELERATION COMPONENTS

$$\mathrm{d}s = \|\mathbf{v}\|\,\mathrm{d}t \qquad \mathbf{T} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = \frac{\mathbf{v}}{\|\mathbf{v}\|} \qquad \mathbf{N} = \frac{\mathrm{d}\mathbf{T}/\mathrm{d}s}{\|\mathrm{d}\mathbf{T}/\mathrm{d}s\|} = \frac{\mathrm{d}\mathbf{T}/\mathrm{d}t}{\|\mathrm{d}\mathbf{T}/\mathrm{d}t\|} \qquad \mathbf{B} = \mathbf{T}\times\mathbf{N}$$

$$\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} = \kappa\mathbf{N} \qquad \frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s} = -\tau\mathbf{N} \qquad \kappa = \left\|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}\right\| = \frac{\|\mathbf{v}\times\mathbf{a}\|}{\|\mathbf{v}\|^3} = \frac{|f''(x)|}{\left\{1+[f'(x)]^2\right\}^{3/2}} = \frac{|\dot{x}\ddot{y}-\dot{y}\ddot{x}|}{(\dot{x}^2+\dot{y}^2)^{3/2}} \qquad \tau = -\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s}\cdot\mathbf{N}$$

$$\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N} \qquad a_T = \frac{\mathrm{d}\|\mathbf{v}\|}{\mathrm{d}t} \qquad a_N = \kappa\|\mathbf{v}\|^2 = \sqrt{\|\mathbf{a}\|^2 - a_T^2}$$

DIRECTIONAL DERIVATIVE, LAGRANGE MULTIPLIERS $\quad D_{\mathbf{u}}f = \dfrac{\mathrm{d}f}{\mathrm{d}s} = \nabla f\cdot\mathbf{u} \qquad \nabla f = \lambda\nabla g, \quad g = 0$

SECOND DERIVATIVES TEST: Suppose $f(x,y)$ and its first and second partial derivatives are continuous in a disk centered at (a,b) and $f_x(a,b) = f_y(a,b) = 0$. Let $D = f_{xx}f_{yy} - f_{xy}^2$.

- If $D > 0$ and $f_{xx} < 0$ at (a,b), then f has a local maximum at (a,b).
- If $D > 0$ and $f_{xx} > 0$ at (a,b), then f has a local minimum at (a,b).
- If $D < 0$ at (a,b), then f has a saddle point at (a,b).
- If $D = 0$ at (a,b), then the test is inconclusive.

TAYLOR'S FORMULA [at the point (x_0,y_0)]

$$f(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + \frac{1}{2!}\left[f_{xx}(x_0,y_0)(x-x_0)^2 + 2f_{xy}(x_0,y_0)(x-x_0)(y-y_0) + f_{yy}(x_0,y_0)(y-y_0)^2\right]$$
$$+ \frac{1}{3!}\left[f_{xxx}(x_0,y_0)(x-x_0)^3 + 3f_{xxy}(x_0,y_0)(x-x_0)^2(y-y_0) + 3f_{xyy}(x_0,y_0)(x-x_0)(y-y_0)^2 + f_{yyy}(x_0,y_0)(y-y_0)^3\right] + \cdots$$

ERROR IN LINEAR APPROXIMATION $\quad |E(x,y)| \le \dfrac{1}{2!}M\left(|x-x_0|+|y-y_0|\right)^2$, where $\max\{|f_{xx}|, |f_{xy}|, |f_{yy}|\} \le M$

ERROR IN QUADRATIC APPROXIMATION $\quad |E(x,y)| \le \dfrac{1}{3!}M\left(|x-x_0|+|y-y_0|\right)^3$, where $\max\{|f_{xxx}|, |f_{xxy}|, |f_{xyy}|, |f_{yyy}|\} \le M$

CHANGE OF VARIABLES/SUBSTITUTIONS IN MULTIPLE INTEGRALS

$$\iint_{\mathcal{R}} f(x,y)\,\mathrm{d}A = \iint_{\mathcal{S}} f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\mathrm{d}u\,\mathrm{d}v \quad \text{where} \quad J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

POLAR COORDINATES $\quad x = r\cos\theta \qquad y = r\sin\theta \qquad r^2 = x^2+y^2 \qquad \mathrm{d}A = \mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$

Coordinate Conversions

Cylindrical to Rectangular	Spherical to Rectangular	Spherical to Cylindrical
$x = r\cos\theta$	$x = \rho\sin\phi\cos\theta$	$r = \rho\sin\phi$
$y = r\sin\theta$	$y = \rho\sin\phi\sin\theta$	$z = \rho\cos\phi$
$z = z$	$z = \rho\cos\phi$	$\theta = \theta$

$$\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z = \rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$$

MASS, MOMENTS, AND CENTER OF MASS $\quad$ Mass $M = \iint_R \delta\,\mathrm{d}A \quad$ Moments $M_x = \iint_R y\,\delta\,\mathrm{d}A \quad M_y = \iint_R x\,\delta\,\mathrm{d}A \quad$ Center of mass $\overline{x} = \dfrac{M_y}{M} \quad \overline{y} = \dfrac{M_x}{M}$

FLOW AND FLUX $\quad$ Flow $= \int_C \mathbf{F}\cdot\mathbf{T}\,\mathrm{d}s = \int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \int_C P\,\mathrm{d}x + Q\,\mathrm{d}y \quad$ Flux $= \int_C \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}s = \int_C P\,\mathrm{d}y - Q\,\mathrm{d}x \quad \mathbf{n} = \mathbf{T}\times\mathbf{k} \quad \mathbf{F}(x,y) = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j}$

GREEN'S THEOREM $\quad \mathcal{C}$ is the boundary curve enclosing the region $\mathcal{D}$, traversed counterclockwise.

FLOW/TANGENTIAL FORM $\quad \iint_{\mathcal{D}}\left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right)\mathrm{d}A = \oint_{\mathcal{C}} P\,\mathrm{d}x + Q\,\mathrm{d}y \qquad$ FLUX/NORMAL FORM $\quad \iint_{\mathcal{D}}\left(\dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y}\right)\mathrm{d}A = \oint_{\mathcal{C}} P\,\mathrm{d}y - Q\,\mathrm{d}x$

SURFACE AREA OF LEVEL SURFACE $g(x,y,z) = c \quad S = \iint_{\mathcal{S}}\mathrm{d}S = \iint_{\mathcal{R}}\dfrac{\|\nabla g\|}{|\nabla g\cdot\mathbf{p}|}\,\mathrm{d}A$

STOKES' THEOREM $\quad \iint_{\mathcal{S}}(\nabla\times\mathbf{F})\cdot\mathrm{d}\mathbf{S} = \oint_{\mathcal{C}}\mathbf{F}\cdot\mathrm{d}\mathbf{r} \qquad \mathcal{C}$ is the boundary curve of the surface $\mathcal{S}$

GAUSS' DIVERGENCE THEOREM $\quad \iiint_{\mathcal{E}}\nabla\cdot\mathbf{F}\,\mathrm{d}V = \iint_{\mathcal{S}}\mathbf{F}\cdot\mathrm{d}\mathbf{S} \qquad \mathcal{S}$ is the boundary surface of the solid $\mathcal{E}$

FUN TRIGONOMETRY FACTS $\quad \sin^2 x = \dfrac{1-\cos 2x}{2} \qquad \cos^2 x = \dfrac{1+\cos 2x}{2}$