NOTE: Any integrals that need to be evaluated will require integration techniques no more complicated than u-substitution or the use of trigonometric identities.

1. [20 pts] Use a change of variables to find the volume of the region under the graph of $f(x,y) = 4xy$ lying over the portion of the first quadrant bounded by the parabolas $x = y^2 - 5$, $x = y^2 - 3$, $x = -y^2 + 6$, and $x = -y^2 + 10$.

SOLUTION:

We need to evaluate $\iint_{\mathcal{R}} 4xy \, \mathrm{d}A$ where $\mathcal{R}$ can be described by the inequalities $-5 \le x - y^2 \le -3$ and $6 \le x + y^2 \le 10$. This suggests the change of variables $u = x - y^2$ and $v = x + y^2$ with the new region of integration, $\mathcal{S}$, $-5 \le u \le -3$ and $6 \le v \le 10$. To compute the Jacobian of the transformation note that $x = \frac{1}{2}(u+v)$ and $y = \sqrt{\frac{1}{2}(v-u)}$ where we need only consider the positive square root since our region is in the first quadrant. Then

$$J(u,v) = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{4\sqrt{\frac{1}{2}(v-u)}} & \frac{1}{4\sqrt{\frac{1}{2}(v-u)}} \end{vmatrix} = \frac{1}{4\sqrt{\frac{1}{2}(v-u)}}$$

and

$$\iint_{\mathcal{R}} 4xy \, \mathrm{d}A = \iint_{\mathcal{S}} 4\left(\frac{1}{2}\right)(u+v)\sqrt{\frac{1}{2}(v-u)}\left|\frac{1}{4\sqrt{\frac{1}{2}(v-u)}}\right| \mathrm{d}u\,\mathrm{d}v = \frac{1}{2}\int_6^{10}\int_{-5}^{-3}(u+v)\,\mathrm{d}u\,\mathrm{d}v$$

$$= \frac{1}{2}\int_6^{10}\left(\frac{u^2}{2} + uv\right)\bigg|_{-5}^{-3}\mathrm{d}v = \frac{1}{2}\int_6^{10}\left[\frac{9}{2} - 3v - \left(\frac{25}{2} - 5v\right)\right]\mathrm{d}v$$

$$= \frac{1}{2}\int_6^{10}(-8+2v)\,\mathrm{d}v = \frac{1}{2}\left(-8v + v^2\right)\bigg|_6^{10} = 16$$

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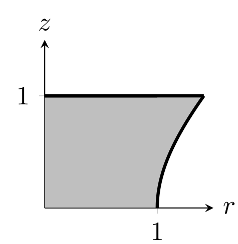
2. [20 pts] The density of soot particles inside a smokestack is given by $\delta(x,y,z) = 12x^2 z$ particles per cubic meter. The smokestack is in the shape of the hyperboloid of one sheet $x^2 + y^2 - z^2 = 1$ for $0 \le z \le 1$. A cross section of half of the smokestack is shown in the accompanying $rz-$plane (constant θ plane). Find the total number (which may not be an integer) of soot particles in the smokestack.



SOLUTION:

If $\mathcal{W}$ is the inside of the smokestack, the total number of particles is equal to $\iiint_{\mathcal{W}} \delta(x,y,z)\,\mathrm{d}V$. Cylindrical coordinates will be helpful with the hyperboloid's equation becoming $r = \sqrt{z^2+1}$ $\left(\text{or } z = \sqrt{r^2-1}\right)$ and the point of intersection of the hyperboloid and the plane $z = 1$ yielding $r = \sqrt{2}$. The density can be written as $\delta(r,\theta,z) = 12r^2 z\cos^2\theta$.

$$\text{Total particles} = \int_0^{2\pi}\int_0^1\int_0^{\sqrt{z^2+1}}\left(12r^2 z\cos^2\theta\right) r\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}\theta = \left(\int_0^{2\pi}\cos^2\theta\,\mathrm{d}\theta\right)\left(12\int_0^1 z\int_0^{\sqrt{z^2+1}} r^3\,\mathrm{d}r\,\mathrm{d}z\right)$$

$$= 12\left[\int_0^{2\pi}\frac{1}{2}(1+\cos 2\theta)\,\mathrm{d}\theta\right]\left(\int_0^1 z\frac{r^4}{4}\bigg|_0^{\sqrt{z^2+1}}\mathrm{d}z\right) = \frac{3}{2}\left[\left(\theta + \frac{1}{2}\sin 2\theta\right)\bigg|_0^{2\pi}\right]\int_0^1 z(z^2+1)^2\,\mathrm{d}z$$

$$= 3\pi\int_0^1\left(z^5 + 2z^3 + z\right)\mathrm{d}z = 3\pi\left(\frac{z^6}{6} + \frac{z^4}{2} + \frac{z^2}{2}\right)\bigg|_0^1 = \frac{7\pi}{2}$$

Alternatively,

$$\begin{aligned}
\text{Total particles} &= \int_0^{2\pi}\int_0^1\int_0^1 \left(12r^2 z\cos^2\theta\right) r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta + \int_0^{2\pi}\int_1^{\sqrt{2}}\int_{\sqrt{r^2-1}}^1 \left(12r^2 z\cos^2\theta\right) r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \left(\int_0^{2\pi}\cos^2\theta\,\mathrm{d}\theta\right)\left(\int_0^1 12r^3\,\mathrm{d}r\right)\left(\int_0^1 z\,\mathrm{d}z\right) + \left(\int_0^{2\pi}\cos^2\theta\,\mathrm{d}\theta\right)\left(\int_1^{\sqrt{2}}\int_{\sqrt{r^2-1}}^1 12r^3 z\,\mathrm{d}z\,\mathrm{d}r\right) \\
&= \left[\int_0^{2\pi}\frac{1}{2}\left(1+\cos 2\theta\right)\,\mathrm{d}\theta\right]\left(3r^4\Big|_0^1\right)\left(\frac{z^2}{2}\Big|_0^1\right) + \left[\int_0^{2\pi}\frac{1}{2}\left(1+\cos 2\theta\right)\,\mathrm{d}\theta\right]\left(\int_1^{\sqrt{2}} 6r^3 z^2\Big|_{\sqrt{r^2-1}}^1\,\mathrm{d}r\right) \\
&= \frac{1}{2}\left(\theta+\frac{1}{2}\sin 2\theta\right)\Big|_0^{2\pi}(3)\left(\frac{1}{2}\right) + \frac{1}{2}\left(\theta+\frac{1}{2}\sin 2\theta\right)\Big|_0^{2\pi}\left\{\int_1^{\sqrt{2}} 6r^3\left[1-(r^2-1)\right]\,\mathrm{d}r\right\} \\
&= \frac{3\pi}{2} + \pi\int_1^{\sqrt{2}}\left(12r^3-6r^5\right)\mathrm{d}r = \frac{3\pi}{2}+\pi\left(3r^4-r^6\right)\Big|_1^{\sqrt{2}} = \frac{7\pi}{2}
\end{aligned}$$

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3. [20 pts] A spherical tank of radius 4 feet is centered at the origin and is partially filled with oil. A straight measuring stick enters the sphere at the point $(\rho,\theta,\phi)=(4,0,0)$ and measures the depth of the oil as 2 feet.

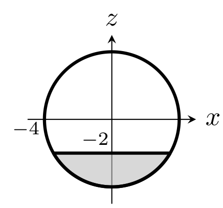
(a) Make a sketch of the situation in the $xz-$plane, being sure to show the location of the oil.

(b) Set up, but **do not** evaluate the integral to find the volume of oil in the tank using the integration order $\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$.

(c) Set up, but **do not** evaluate the integral to find the volume of oil in the tank using the integration order $\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$.

SOLUTION:

(a) Sketch of region



(b) The plane $z=-2$ intersects the sphere when $x^2+y^2+(-2)^2=16 \implies x^2+y^2=12$. Thus,

$$\text{Volume} = \int_{-2\sqrt{3}}^{2\sqrt{3}}\int_{-\sqrt{12-y^2}}^{\sqrt{12-y^2}}\int_{-\sqrt{16-x^2-y^2}}^{-2}\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$$

(c) The top of the oil is the plane $z=-2 \implies \rho=-2\sec\phi$. The plane intersects the sphere when $-2\sec\phi=4 \implies \phi=\frac{2\pi}{3}$.

$$\text{Volume} = \int_0^{2\pi}\int_{2\pi/3}^{\pi}\int_{-2\sec\phi}^{4}\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$$

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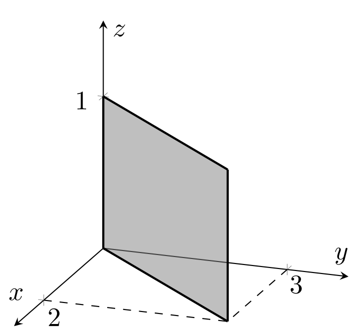
4. [20 pts] A thin metal rectangular plate is standing vertically in the first octant as shown. If the density of the metal is $1+6x+8y(z+1)$ pounds per square foot, how much does the plate weigh?



SOLUTION:

We need to compute the surface integral of the density. The plate is a portion of the plane $2y - 3x = 0$, so the surface over which the integration takes place is $g(x,y,z) = 2y - 3x \implies \nabla g = \langle -3, 2, 0\rangle \implies \|\nabla g\| = \sqrt{13}$. We can project onto either the $yz-$plane or $xz-$plane.

projection plane	yz	xz
$\mathbf{p}$	$\mathbf{i}$	$\mathbf{j}$
$\lvert\nabla g \cdot \mathbf{p}\rvert$	$\lvert -3\rvert = 3$	$\lvert 2\rvert = 2$
$\mathcal{R}$	$0 \le y \le 3, 0 \le z \le 1$	$0 \le x \le 2, 0 \le z \le 1$
set up	$\iint_{\mathcal{R}} [1 + 6x + 8y(z+1)] \frac{\sqrt{13}}{3} \,\mathrm{d}y\,\mathrm{d}z$	$\iint_{\mathcal{R}} [1 + 6x + 8y(z+1)] \frac{\sqrt{13}}{2} \,\mathrm{d}x\,\mathrm{d}z$
eliminate variables	$x = \frac{2}{3}y$	$y = \frac{3}{2}x$
weight	$\frac{\sqrt{13}}{3}\int_0^1\int_0^3 (1 + 12y + 8yz)\,\mathrm{d}y\,\mathrm{d}z$	$\frac{\sqrt{13}}{2}\int_0^1\int_0^2 (1 + 18x + 12xz)\,\mathrm{d}x\,\mathrm{d}z$
	$= \frac{\sqrt{13}}{3}\int_0^1 \left(y + 6y^2 + 4y^2 z\right)\Big\rvert_0^3 \mathrm{d}z$	$= \frac{\sqrt{13}}{2}\int_0^1 \left(x + 9x^2 + 6x^2 z\right)\Big\rvert_0^2 \mathrm{d}z$
	$= \frac{\sqrt{13}}{3}\int_0^1 (57 + 36z)\,\mathrm{d}z$	$= \frac{\sqrt{13}}{2}\int_0^1 (38 + 24z)\,\mathrm{d}z$
	$= \frac{\sqrt{13}}{3}\left(57z + 18z^2\right)\Big\rvert_0^1$	$= \frac{\sqrt{13}}{2}\left(38z + 12z^2\right)\Big\rvert_0^1$
	$= 25\sqrt{13}$	$= 25\sqrt{13}$

The plate weighs $25\sqrt{13}$ pounds. ■

5. [20 pts] You have a cool new "exponential" curtain covering a window in your apartment. It touches the floor ($xy-$plane) along the curve $y = e^{x/2}$ lying between $x = \ln 32$ and $x = \ln 60$. The top of the curtain reaches up to the surface $z = 3e^x$. All units are in feet. The fabric making up the curtain is sensitive to sunlight so you need to purchase some UV protection to spray on the side of the curtain facing the window. The UV protector comes in spray cans that each cover 100 square feet. Use a line integral to determine how many full cans of the spray you must purchase to protect your curtain.

SOLUTION:

We need to find the area of one side of the curtain, given by $\displaystyle\int_{\mathcal{C}} 3e^x\,\mathrm{d}s$. The base of the curtain, the function $y = e^{x/2}$, can be parameterized as

$$\mathbf{r}(t) = \left\langle t, e^{t/2}\right\rangle \quad \ln 32 \le t \le \ln 60$$

$$\implies \mathbf{r}'(t) = \left\langle 1, \tfrac{1}{2}e^{t/2}\right\rangle \implies \|\mathbf{r}'(t)\| = \sqrt{1 + \left(\tfrac{1}{2}e^{t/2}\right)^2} = \sqrt{1 + \tfrac{1}{4}e^t}$$

Then

$$\int_{\mathcal{C}} 3e^x\,\mathrm{d}s = \int_{\ln 32}^{\ln 60} 3e^t\sqrt{1 + \tfrac{1}{4}e^t}\,\mathrm{d}t$$

$$u = 1 + \tfrac{1}{4}e^t, \quad \mathrm{d}u = \tfrac{1}{4}e^t\,\mathrm{d}t; \quad t = \ln 32 \implies u = 9; \quad t = \ln 60 \implies u = 16$$

$$\int_{\mathcal{C}} 3e^x\,\mathrm{d}s = 12\int_9^{16} u^{1/2}\,\mathrm{d}u = 8u^{3/2}\Big\rvert_9^{16} = 8(64 - 27) = 296 \text{ ft}^2$$

Since each can of UV protection covers 100 square feet, we need to purchase 3 cans. ■