1. [27 pts] You and a group of friends are hiking in Calcthree Municipal Park. The park's boundaries are given by $|x| = 2$ and $|y| = 2$ and the elevation above sea level in the park is described by the function $z(x, y) = 10 + x^3 - 3x + 2y^2 - y^4$.

   (a) [3 pts] Is there a highest point in the park to which you and your friends can hike? Don't try to find it, just decide if there is one and in a single sentence justify your answer.

   (b) [24 pts] Provide mathematical justifications for your answers to the following questions:

      i. [6 pts] You are standing at the point $(1, -1, 9)$. Are you in a valley, on a hilltop, in a saddle or none of these?

      ii. [6 pts] You spot a bear whose (x, y) coordinates are $(-1, 1)$. Is it above or below you? Is it on a hilltop?

      iii. [6 pts] Your friends are standing at the location in the park that is directly above the origin. They also spot the bear. To avoid it they decide to head toward your location. When they start out, will they be ascending or descending? At what rate?

      iv. [6 pts] If your friends want to lose elevation at the fastest rate, in which direction should they begin walking from the origin to accomplish this? Write your answer as a unit vector.

SOLUTION:

(a) Yes, the elevation function is continuous and the park is defined on a closed, bounded set. The Extreme Value Theorem guarantees an absolute maximum of the function.

(b) We need to find the critical points of the function.

$$\frac{\partial z}{\partial x} = 3x^2 - 3 \qquad \frac{\partial^2 z}{\partial x^2} = 6x$$

$$\frac{\partial z}{\partial y} = 4y - 4y^3 \qquad \frac{\partial^2 z}{\partial y^2} = 4 - 12y^2$$

$$\frac{\partial^2 z}{\partial x \partial y} = 0 \qquad D(x, y) = (6x)(4 - 12y^2)$$

$$3x^2 - 3 = 3(x + 1)(x - 1) = 0 \implies x = -1, 1$$

$$4y - 4y^3 = 4(y)(1 - y)(1 + y) = 0 \implies y = -1, 0, 1$$

i. $D(1, -1) = (6)(1)\left[4 - 12(-1)^2\right] = -48 < 0 \implies$ You are standing in a saddle

ii.

$$z(-1, 1) = 10 + (-1)^3 - 3(-1) + 2(1)^2 - 1^4 = 13$$

$$D(-1, 1) = 6(-1)\left[4 - 12(-1)^2\right] = 48 > 0, \qquad z_{xx}(-1, 1) = 6(-1) = -6 < 0 \implies \text{local maximum}$$

The bear is standing on a hilltop at an elevation of 13 so it is above you since your elevation is 9.

iii. A unit vector in the direction your friends need to walk is $\mathbf{u} = \frac{1}{\sqrt{2}}\,\mathbf{i} - \frac{1}{\sqrt{2}}\,\mathbf{j}$. With $\nabla z(0, 0) = -3\,\mathbf{i}$,

$$D_{\mathbf{u}} z(0, 0) = \left.\frac{\mathrm{d}z}{\mathrm{d}s}\right|_{(0,0)} = \langle -3, 0\rangle \cdot \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle = -\frac{3\sqrt{2}}{2}$$

Your friends will be descending at a rate of $-3\sqrt{2}/2$.

iv. Your friends need to go in the direction of $-\nabla z(0, 0) = 3\,\mathbf{i}$ implying that they should walk in the $3\,\mathbf{i}$ direction to begin descending at the fastest rate. A unit vector in this direction is simply $\mathbf{i}$.

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2. [28 pts] Let $g(x, y) = \sqrt{4 - 3x + 2y}$.

   (a) [6 pts] Find and sketch the domain of $g(x, y)$. Label any intercepts.

   (b) [2 pts] Find the range of $g(x, y)$, writing your answer using interval notation.

   (c) [3 pts] Where is $g(x, y)$ continuous?

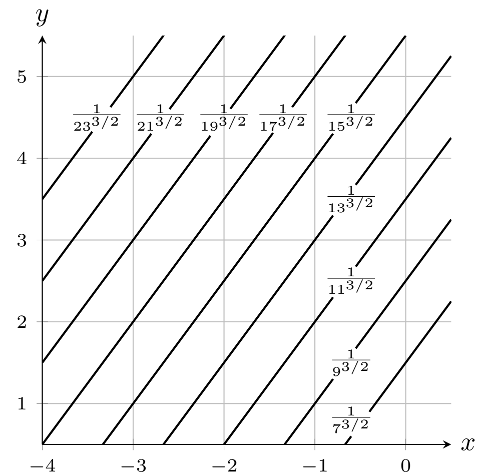
   (d) [3 pts] Find $\lim_{(x,y)\to(0,0)} g(x, y)$ or show that is does not exist.

   (e) [8 pts] Find the first order Taylor polynomial (linearization) for $g(x, y)$ centered at $(-2, 3)$. Write your final answer in the form $T_1(x, y) = ax + by + c$.

(f) [6 pts] The following figure shows a contour map of the function $h(x, y) = (4 - 3x + 2y)^{-3/2}$. Use this and the fact that

$$g_{xx}(x, y) = \frac{-9}{4(4 - 3x + 2y)^{3/2}} \qquad g_{xy}(x, y) = \frac{-3}{2(4 - 3x + 2y)^{3/2}} \qquad g_{yy}(x, y) = \frac{-1}{(4 - 3x + 2y)^{3/2}}$$

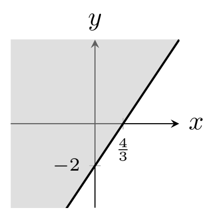
to find an upper bound on the absolute value of the error in the linearization of $g(x, y)$ if $|x + 2| < 1$ and $|y - 3| < 2$.



SOLUTION:

(a)

$$4 - 3x + 2y \geq 0 \implies y \geq \frac{3}{2}x - 2 \implies \text{domain of } g(x, y) \text{ is } \left\{ (x, y) \in \mathbb{R}^2 \,\middle|\, y \geq \frac{3}{2}x - 2 \right\}$$



(b) $[0, \infty)$

(c) $g(x, y)$ is continuous on its domain

(d)

$$\lim_{(x,y)\to(0,0)} g(x, y) = \lim_{(x,y)\to(0,0)} \sqrt{4 - 3x + 2y} = \sqrt{4 - 3(0) + 2(0)} = 2$$

(e)

$$g(-2, 3) = \sqrt{4 - 3(-2) + 3(2)} = 4$$

$$g_x(x, y) = \frac{-3}{2\sqrt{4 - 3x + 2y}} \implies g_x(-2, 3) = \frac{-3}{2\sqrt{4 - 3(-2) + 3(2)}} = -\frac{3}{8}$$

$$g_y(x, y) = \frac{1}{\sqrt{4 - 3x + 2y}} \implies g_y(-2, 3) = \frac{1}{\sqrt{4 - 3(-2) + 3(2)}} = \frac{1}{4}$$

$$T_1(x, y) = 4 - \frac{3}{8}(x + 2) + \frac{1}{4}(y - 3) = \frac{5}{2} - \frac{3}{8}x + \frac{1}{4}y$$

(f) The largest value attained by $h(x, y)$ on the rectangle of interest is $9^{-3/2}$. Thus

$$|g_{xx}(x, y)| = \left| -\frac{9}{4}h(x, y) \right| \leq \frac{9}{4}\left(\frac{1}{9^{3/2}}\right) = \frac{1}{12}$$

$$|g_{xy}(x, y)| = \left| -\frac{3}{2}h(x, y) \right| \leq \frac{3}{2}\left(\frac{1}{9^{3/2}}\right) = \frac{1}{18}$$

$$|g_{yy}(x, y)| = |-h(x, y)| \leq \frac{1}{9^{3/2}} = \frac{1}{27}$$

$$\implies M = \frac{1}{12} \implies |E(x, y)| \leq \frac{1/12}{2}(1 + 2)^2 = \frac{3}{8}$$

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3. [20 pts] The temperature in a region of space is given by $T(x, y, z) = 1000 + x^2 + y^2 + z^2$. The super-duper Rate-O-Change meter on board your spaceship gives a readout of the instantaneous rate of change of temperature with respect to any variable you enter into it. To receive credit, you must use Calculus 3 concepts to answer this question. Improper notation will be penalized.

   (a) [10 pts] Suppose you know the following information about the path of your spaceship:
   $$\mathbf{r}(0) = \mathbf{k} \qquad \mathbf{r}(1) = 2\,\mathbf{i} + \mathbf{k} \qquad \mathbf{r}'(0) = -4\,\mathbf{i} + 2\,\mathbf{j} + 7\,\mathbf{k} \qquad \mathbf{r}'(1) = 2\,\mathbf{i} + 2\pi\,\mathbf{j}$$
   What does your meter read when you enter t into it and you are at the point $(2, 0, 1)$?

   (b) [10 pts] Now suppose your spaceship's position is given by $x(v, w) = (v^2 + w^2)$, $y(u, v) = \ln(uv)$, $z(u, v) = e^{2u+4v}$. What does your meter read if you enter u into it when $u = 2, v = \frac{1}{2}$ and $w = 1$?

SOLUTION:

(a) We are at the point $(2, 0, 1)$ when $t = 1$.
$$\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{\partial T}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial T}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial T}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} = \nabla T[\mathbf{r}(t)] \cdot \mathbf{r}'(t)$$
$$\Longrightarrow \left.\frac{\mathrm{d}T}{\mathrm{d}t}\right|_{t=1} = \nabla T\left[\mathbf{r}(1)\right] \cdot \mathbf{r}'(1) = \left.\langle 2x, 2y, 2z\rangle\right|_{(2,0,1)} \cdot \mathbf{r}'(1) = \langle 2(2), 2(0), 2(1)\rangle \cdot \langle 2, 2\pi, 0\rangle = 8$$

(b) When $u = 2, v = \frac{1}{2}$ and $w = 1$, $x = \frac{1}{4} + 1 = \frac{5}{4}$, $y = \ln\left[(2)\left(\frac{1}{2}\right)\right] = \ln 1 = 0$, $z = e^{2(2)+4(1/2)} = e^6$
$$\frac{\partial T}{\partial u} = \frac{\partial T}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial T}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial T}{\partial z}\frac{\partial z}{\partial u} = 2x(0) + 2y\left(\frac{v}{uv}\right) + 2z(2e^{2u+4v}) = \frac{2y}{u} + 4ze^{2u+4v}$$
$$\left.\frac{\partial T}{\partial u}\right|_{(u,v,w)=\left(2,\frac{1}{2},1\right)} = \frac{2(0)}{2} + 4e^6e^6 = 4e^{12}$$

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4. [10 pts] The body mass index (B) as a function of weight (W, kg) and height (H, m) is given by $B = W/H^2$. Suppose that for a 2 m tall, 100 kg person you know that the height measurement is 0.01 m too high. If you want the body mass index to have an error no greater than 0.25, use differentials to determine the maximum error that can be present in the weight measurement.

SOLUTION:

The differential of B is
$$\mathrm{d}B = \frac{\partial B}{\partial W}\mathrm{d}W + \frac{\partial B}{\partial H}\mathrm{d}H = \frac{1}{H^2}\mathrm{d}W - \frac{2W}{H^3}\mathrm{d}H$$
Applying this at the point $(W, H) = (100, 2)$, noting that $\mathrm{d}H = 0.01$ and requiring $\mathrm{d}B$ to be bounded above by 0.25 yields
$$\mathrm{d}B = \frac{1}{4}\mathrm{d}W - \frac{(2)(100)}{2^3}\left(\frac{1}{100}\right) = \frac{1}{4}\mathrm{d}W - \frac{1}{4} \leq \frac{1}{4} \Longrightarrow \mathrm{d}W \leq 2$$
The maximum error that can be present in the weight measurement is 2 kg. ■

5. [15 pts] The satisfaction you get from eating x slices of bacon, y eggs and z potatoes is given by the satisfaction function $s(x, y, z) = \sqrt{xyz}$. Suppose you have six dollars to spend and a rather hearty appetite. If each slice of bacon costs 50 cents, each egg costs 25 cents and each potato costs 10 cents, determine how many bacon slices, eggs and potatoes you should purchase to be the most satisfied. You must use Calculus III concepts to obtain any credit on this problem.

SOLUTION:

We need to optimize $s(x, y, z)$ subject to the constraint $g(x, y, z) = \frac{1}{2}x + \frac{1}{4}y + \frac{1}{10}z = 6$ with x, y, z all nonnegative. We solve using the Lagrange multipliers technique.
$$s_x = \frac{yz}{2\sqrt{xyz}} = \frac{\sqrt{yz}}{2\sqrt{x}} \qquad g_x = \frac{1}{2}$$
$$s_y = \frac{xz}{2\sqrt{xyz}} = \frac{\sqrt{xz}}{2\sqrt{y}} \qquad g_y = \frac{1}{4}$$
$$s_z = \frac{xy}{2\sqrt{xyz}} = \frac{\sqrt{xy}}{2\sqrt{z}} \qquad g_z = \frac{1}{10}$$

yielding the following system of nonlinear equations

$$\frac{\sqrt{yz}}{2\sqrt{x}} = \frac{1}{2}\lambda \implies \frac{\sqrt{yz}}{\sqrt{x}} = \lambda \tag{1}$$

$$\frac{\sqrt{xz}}{2\sqrt{y}} = \frac{1}{4}\lambda \implies \frac{2\sqrt{xz}}{\sqrt{y}} = \lambda \tag{2}$$

$$\frac{\sqrt{xy}}{2\sqrt{z}} = \frac{1}{10}\lambda \implies \frac{5\sqrt{xy}}{\sqrt{z}} = \lambda \tag{3}$$

$$\frac{1}{2}x + \frac{1}{4}y + \frac{1}{10}z = 6 \tag{4}$$

Note that none of x, y or z can be zero in this system. Combining (1) and (2) yields

$$\frac{\sqrt{yz}}{\sqrt{x}} = \frac{2\sqrt{xz}}{\sqrt{y}} \implies y = 2x$$

while combining (1) and (3) yields

$$\frac{\sqrt{yz}}{\sqrt{x}} = \frac{5\sqrt{xy}}{\sqrt{z}} \implies z = 5x$$

Using this information in (4) gives

$$\frac{1}{2}x + \frac{1}{4}(2x) + \frac{1}{10}(5x) = 6 \implies \frac{3}{2}x = 6 \implies x = 4$$

from which we conclude that $y = 8$ and $z = 20$. To maximize our satisfaction we should buy 4 slices of bacon, 8 eggs and 20 potatoes.

Note that we are maximizing the function on a closed, bounded set (that portion of a plane in the first octant). Since the function is continuous on this set, extrema will occur either at interior critical points or on the boundary. The only critical point is $(4, 8, 20)$ (interior to the set) for which $s(x, y, z) = 8\sqrt{10}$. The function vanishes on the boundary so we conclude that this indeed the maximum value of the satisfaction function. ■