1. [12 pts] In your bluebook, write the word **TRUE** if the statement is true or write the word **FALSE** if the statement is false. No justification needed and no partial credit given.

   (a) If the velocity and acceleration vectors of a curved path are both nonzero, but scalar multiples of one another at a point, then the normal component of the acceleration is zero at that point.

   (b) $-x^2 + y^2 - 2z^2 + 2x + 2y + 4z - 1 = 0$ represents an elliptic cone with vertex at $(-1, 1, -1)$.

   (c) For any two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$, $\text{comp}_{\mathbf{a}}\mathbf{b} = (\text{proj}_{\mathbf{a}}\mathbf{b})\,\mathbf{a}$

   (d) A particle moving along the path $\mathbf{r}(t) = (-4 - \sin 2t)\,\mathbf{i} - \mathbf{j} + (5 + \cos 2t)\,\mathbf{k}$ experiences constant curvature.

   SOLUTION:

   (a) **TRUE** $\mathbf{a}_N$ is proportional to the cross product of the velocity and acceleration vectors, which is zero for vectors that are scalar multiples of one another.

   (b) **FALSE** Complete the square: $(x-1)^2 - (y+1)^2 + 2(z-1)^2 = 1$; hyperboloid of one sheet

   (c) **FALSE** $\text{proj}_{\mathbf{a}}\mathbf{b} = (\text{comp}_{\mathbf{a}}\mathbf{b})\,\dfrac{\mathbf{a}}{\|\mathbf{a}\|}$

   (d) **TRUE** The path is a circle, the curvature of which is a constant (reciprocal of the radius), which is 1.

■

2. [30 pts] A mosquito is buzzing along the path $\mathbf{r}(t) = t\,\mathbf{i} + \frac{2\sqrt{2}}{3}t^{3/2}\,\mathbf{j} + \frac{1}{2}t^2\,\mathbf{k},\ t \geq 0$ with distance measured in yards and time in seconds.

   (a) [10 pts] Starting from time $t = 0$, how long does it take the mosquito to travel 12 yards along this path?

   (b) [10 pts] After 1 second, how far is the mosquito from its starting position at $t = 0$?

   (c) [10 pts] Now suppose that after flying along the aforementioned path for 2 seconds, the mosquito notices a bat approaching from behind. In an effort to avoid becoming the bat's dinner, from this point the mosquito leaves the original path, flying straight ahead along a line at a constant speed of 1 yard/sec. Find the mosquito's coordinates after it travels along this line for 3 seconds.

   SOLUTION:

   (a)
$$\mathbf{r}'(t) = \mathbf{i} + \sqrt{2t}\,\mathbf{j} + t\,\mathbf{k} \implies \|\mathbf{r}'(t)\| = \sqrt{1 + 2t + t^2} = \sqrt{(1+t)^2} = |1+t| = 1+t \ \text{ since } t \geq 0$$
$$L = 12 = \int_0^c \|\mathbf{r}'(t)\|\,\mathrm{d}t = \int_0^c (1+t)\,\mathrm{d}t = \left(t + \frac{t^2}{2}\right)\Bigg|_0^c = c + \frac{c^2}{2}$$
$$\implies c^2 + 2c - 24 = (t+6)(t-4) = 0 \implies t = -6, 4$$

   Since we are only interested in positive t values, it takes the mosquito 4 seconds to fly 12 yards

   (b) After 1 second, the mosquito is at the point $\left(1, \frac{2\sqrt{2}}{3}, \frac{1}{2}\right)$. It was initially at the origin so the distance from the starting point is
$$d = \sqrt{1^2 + \left(\frac{2\sqrt{2}}{3}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{1 + \frac{8}{9} + \frac{1}{4}} = \sqrt{\frac{36 + 32 + 9}{36}} = \frac{\sqrt{77}}{6} \text{ yards}$$

   (c) After 2 seconds, the mosquito's position vector is $\mathbf{r}(2) = 2\,\mathbf{i} + \frac{8}{3}\,\mathbf{j} + 2\,\mathbf{k}$ and the tangent vector there is $\mathbf{r}'(2) = \mathbf{i} + 2\,\mathbf{j} + 2\,\mathbf{k}$. Since the speed must be one, we find a unit vector in the direction of the tangent vector and use the arclength parameterization of the line to obtain
$$\mathbf{r}_{\text{line}}(s) = \left\langle 2, \tfrac{8}{3}, 2 \right\rangle + s\frac{\langle 1, 2, 2\rangle}{\sqrt{1^2 + 2^2 + 2^2}} = \left\langle 2, \tfrac{8}{3}, 2 \right\rangle + \frac{s}{3}\langle 1, 2, 2\rangle$$

   After 3 seconds flying on the line, the mosquito's position vector is (coordinates are)
$$\mathbf{r}_{\text{line}}(3) = \left\langle 2, \tfrac{8}{3}, 2 \right\rangle + \tfrac{3}{3}\langle 1, 2, 2\rangle = \left\langle 3, \tfrac{14}{3}, 4 \right\rangle \ \text{ or } \ \left(3, \tfrac{14}{3}, 4\right)$$

■

3. [28 pts] Consider the path given by $\mathbf{r}(t) = \sqrt{3}\,t\,\mathbf{i} + \sin t\,\mathbf{j} + \cos t\,\mathbf{k},\ \ t \geq 0$.

   (a) [14 pts] Find the value(s) of t in the interval $[0, 2\pi]$ where the osculating plane is parallel to the plane $-\frac{1}{2}x + \frac{3}{4}y - \frac{\sqrt{3}}{4}z = 1$.

(b) [14 pts] Find the equation of the normal plane to the path when $t = \pi/6$. Write your answer in the form $ax + by + cz = d$.

SOLUTION:

(a) The osculating plane will be parallel to the given plane when the osculating plane's normal vector (the binormal vector, $\mathbf{B}$) is parallel to the given plane's normal, $\mathbf{n} = \left\langle -\frac{1}{2}, \frac{3}{4}, -\frac{\sqrt{3}}{4} \right\rangle$.

$$\mathbf{r}'(t) = \sqrt{3}\,\mathbf{i} + \cos t\,\mathbf{j} - \sin t\,\mathbf{k} \implies \|\mathbf{r}'(t)\| = \sqrt{3 + \cos^2 t + \sin^2 t} = 2 \implies \mathbf{T}(t) = \frac{\sqrt{3}}{2}\,\mathbf{i} + \frac{1}{2}\cos t\,\mathbf{j} - \frac{1}{2}\sin t\,\mathbf{k}$$

$$\mathbf{T}'(t) = -\frac{1}{2}\sin t\,\mathbf{j} - \frac{1}{2}\cos t\,\mathbf{k} \implies \|\mathbf{T}'(t)\| = \sqrt{\frac{1}{4}\sin^2 t + \frac{1}{4}\cos^2 t} = \frac{1}{2} \implies \mathbf{N}(t) = -\sin t\,\mathbf{j} - \cos t\,\mathbf{k}$$

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\sqrt{3}}{2} & \frac{1}{2}\cos t & -\frac{1}{2}\sin t \\ 0 & -\sin t & -\cos t \end{vmatrix} = -\frac{1}{2}\,\mathbf{i} + \frac{\sqrt{3}}{2}\cos t\,\mathbf{j} - \frac{\sqrt{3}}{2}\sin t\,\mathbf{k}$$

We require

$$\left.\begin{array}{l} \frac{\sqrt{3}}{2}\cos t = \frac{3}{4} \implies \cos t = \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2}\sin t = -\frac{\sqrt{3}}{4} \implies \sin t = \frac{1}{2} \end{array}\right\} \implies t = \frac{\pi}{6}$$

(b) To find the equation of a plane we need a normal vector to the plane, $\mathbf{T}(\pi/6)$, and point in the plane, $\mathbf{r}(\pi/6)$.

$$\mathbf{T}(\pi/6) = \frac{\sqrt{3}}{2}\,\mathbf{i} + \frac{1}{2}\cos\frac{\pi}{6}\,\mathbf{j} - \frac{1}{2}\sin\frac{\pi}{6}\,\mathbf{k} = \frac{\sqrt{3}}{2}\,\mathbf{i} + \frac{\sqrt{3}}{4}\,\mathbf{j} - \frac{1}{4}\,\mathbf{k}$$

$$\mathbf{r}(\pi/6) = \frac{\sqrt{3}\pi}{6}\,\mathbf{i} + \sin\frac{\pi}{6}\,\mathbf{j} + \cos\frac{\pi}{6}\,\mathbf{k} = \frac{\sqrt{3}\pi}{6}\,\mathbf{i} + \frac{1}{2}\,\mathbf{j} + \frac{\sqrt{3}}{2}\,\mathbf{k}$$

The equation of the plane is then

$$\frac{\sqrt{3}}{2}\left(x - \frac{\sqrt{3}\pi}{6}\right) + \frac{\sqrt{3}}{4}\left(y - \frac{1}{2}\right) - \frac{1}{4}\left(z - \frac{\sqrt{3}}{2}\right) = 0$$

$$2\sqrt{3}x - \pi + \sqrt{3}y - \frac{\sqrt{3}}{2} - z + \frac{\sqrt{3}}{2} = 0$$

$$2\sqrt{3}x + \sqrt{3}y - z = \pi$$

■

4. [30 pts] You are playing indoors with a paper airplane, the velocity of which is given by $\mathbf{v}(t) = (1 - 3\cos t)\,\mathbf{i} + 3\sin t\,\mathbf{k}$.

(a) [10 pts] Find the tangential component of the acceleration vector.

(b) [10 pts] If the initial position of the airplane is $\mathbf{r}(0) = \mathbf{i} - \mathbf{j} + \mathbf{k}$, find an expression for the path of the airplane.

(c) [10 pts] Look out! On its maiden voyage the airplane crashes into the ceiling, considered as a plane parallel to the $xy-$plane. The crash occurs when $t = \pi/2$. Find the acute angle at which the airplane hits the ceiling.

SOLUTION:

(a)

$$a_T = \frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}\| = \frac{\mathrm{d}}{\mathrm{d}t}\sqrt{1 - 6\cos t + 9\cos^2 t + 9\sin^2 t} = \frac{\mathrm{d}}{\mathrm{d}t}\sqrt{10 - 6\cos t} = \frac{3\sin t}{\sqrt{10 - 6\cos t}}$$

(b)

$$\mathbf{r}(t) = \int \mathbf{v}(t)\,\mathrm{d}t = \left[\int (1 - 3\cos t)\,\mathrm{d}t\right]\mathbf{i} + \left[\int 3\sin t\,\mathrm{d}t\right]\mathbf{k} = (t - 3\sin t + c_1)\,\mathbf{i} + c_2\,\mathbf{j} + (-3\cos t + c_3)\,\mathbf{k}$$

$$\mathbf{r}(0) = c_1\,\mathbf{i} + c_2\,\mathbf{j} + (-3 + c_3)\,\mathbf{k} = \mathbf{i} - \mathbf{j} + \mathbf{k} \implies c_1 = 1,\ c_2 = -1,\ c_3 = 4$$

$$\implies \mathbf{r}(t) = (t - 3\sin t + 1)\,\mathbf{i} - \mathbf{j} + (4 - 3\cos t)\,\mathbf{k}$$

(c) A vector normal to the ceiling is $\mathbf{k}$ and a vector in the direction of motion at the point of impact is

$$\mathbf{r}'(\pi/2) = \mathbf{v}(\pi/2) = \mathbf{i} + 3\,\mathbf{k}$$

giving the angle between these as

$$\theta = \cos^{-1}\frac{\mathbf{v}(\pi/2)\cdot\mathbf{k}}{\|\mathbf{v}(\pi/2)\|\|\mathbf{k}\|} = \cos^{-1}\frac{3}{\sqrt{10}}$$

The the angle at which the airplane hits the ceiling is $\dfrac{\pi}{2} - \cos^{-1}\dfrac{3}{\sqrt{10}}$. Alternatively, using $-\mathbf{k}$ as the normal to the ceiling yields the angle of impact as $\cos^{-1}\dfrac{-3}{\sqrt{10}} - \dfrac{\pi}{2}$.

■