APPM 2350—Final Exam 3—285 points

Monday December 17, 7:30am–10am, 2018

ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your student ID number, (3) lecture section/time (4) your instructor's name, and (5) a grading table. Text books, class notes, and calculators are NOT permitted. A one-page one-sided crib sheet is allowed.

Problem 1 – True/False: (30 points)
For the following true/false questions, write TRUE (for always true) or FALSE (if not always true). Your work will not be graded.

(a) The function

$$f(x,y) = \begin{cases} \dfrac{e^{x^2+y^2}-1}{5(x^2+y^2)} & \text{if } (x,y) \neq (0,0) \\ \dfrac{1}{5} & \text{if } (x,y) = (0,0) \end{cases}$$

is a continuous function.

(b) The function $u(x,t) = \text{sech}\,(x-t)$ is a solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}.$$

(c) Suppose $f(x,y)$ has continuous first partial derivatives. Then the directional derivative of f in the direction of the gradient vector ∇f is always greater than or equal to zero.

(d) The following limit exists

$$\lim_{(x,y)\to(0,0)} \frac{|x|}{|x|+|y|}.$$

(e) Suppose $\rho(x,y)$ is the mass density of a lamina (thin, flat, two-dimensional material) that occupies a finite, simple region R contained in $\mathbb{R}^2$. Then the lamina's total mass is $\rho_{\text{avg}}A$ where A is the lamina's area.

(f) The vector field $\mathbf{F}(x,y,z) = \langle \sin y, x\cos y + \cos z, -y\sin z\rangle$ is conservative.

<u>**SOLUTION:**</u>

(a) TRUE. Converting to polar coordinates, note that

$$\begin{aligned} \lim_{(x,y)\to(0,0)} f(x,y) &= \lim_{r\to 0^+} \frac{e^{r^2}-1}{5r^2} && \text{indeterminate } \frac{0}{0} \\ &= \lim_{r\to 0^+} \frac{2re^{r^2}}{10r} && \text{L'Hôpital's Rule} \\ &= \frac{1}{5}\lim_{r\to 0^+} e^{r^2} \\ &= \frac{1}{5}, \end{aligned}$$

so the function is continuous at $(0,0)$.

(b) TRUE. $u(x,t)$ can be written in the form $u(x,t) = f(x-t)$ where $f(z) = \text{sech}\,(z)$. We let $z(x,t) = x-t$ so that $u(x,t) = f(z(x,t))$ where $z_x = 1$ and $z_t = -1$. Then, applying the chain rule, we find

$$\begin{aligned} u_t(x,t) &= f'(z)z_t = -f'(z), \quad & u_{tt}(x,t) &= -f''(z)z_t = f''(z), \\ u_x(x,t) &= f'(z)z_x = f'(z), \quad & u_{xx}(x,t) &= f''(z)z_x = f''(z), \end{aligned}$$

so that $u_{tt} - u_{xx} = f''(z) - f''(z) = f''(x-t) - f''(x-t) = 0$.

(c) TRUE. The directional derivative of f in the direction $\mathbf{u}$ is $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u}$. If $\nabla f = \mathbf{0}$, then $D_{\mathbf{u}}f = 0$ for any $\mathbf{u}$. Suppose $\nabla f \neq \mathbf{0}$, then substituting $\mathbf{u} = \nabla f/|\nabla f|$, we have $D_{\mathbf{u}}f = \nabla f \cdot (\nabla f/|\nabla f|) = |\nabla f| > 0$ and the result is proved.

(d) FALSE. Approach $(0,0)$ along x–axis ($y=0$) yields $\lim_{(x,0)\to(0,0)} \frac{|x|}{|x|+|0|} = 1$. Approach (0,0) along y–axis ($x=0$) yields $\lim_{(0,y)\to(0,0)} \frac{|0|}{|0|+|y|} = 0$. Two approaches to $(0,0)$ yield two different limits, thus limit does not exist.

(e) TRUE. The average value of the function $\rho(x,y)$ over R is

$$\rho_{\text{avg}} = \frac{\iint_R \rho(x,y)\,\mathrm{d}x\mathrm{d}y}{\iint_R \mathrm{d}x\mathrm{d}y} = \frac{\iint_R \rho(x,y)\,\mathrm{d}x\mathrm{d}y}{\text{Area}(R)}.$$

The total mass of the lamina is

$$M = \iint_R \rho(x,y)\,\mathrm{d}x\mathrm{d}y = \rho_{\text{avg}}\text{Area}(R).$$

(f) TRUE. Let $\mathbf{F} = \langle P, Q, R\rangle = \langle \sin y, x\cos y + \cos z, -y\sin z\rangle$, then we check the conditions for a conservative vector field

$$\frac{\partial P}{\partial y} = \cos y = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z} = 0 = \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial z} = -\sin z = \frac{\partial R}{\partial y}.$$

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Problem 2 – Short Answer Questions: (90 points)
For the questions in this problem, show all your work and clearly $\boxed{\text{box}}$ your final answer. Partial credit may be given.

(a) (25 pts) Suppose the position of a particle at time t is given by the position vector $\mathbf{r}(t) = \langle 1, \cos^2 t, \sin^2 t\rangle$.
  (i) Find the particle's velocity and acceleration vectors.
  (ii) What distance did the particle travel from $t=0$ to $t=\pi$?

(b) (25 pts) Consider the function $f(x,y) = x^2 + 2y^2 - x^2 y$.
  (i) Find the critical points of the function.
  (ii) Classify the critical points of the function as either a local maximum, local minimum or saddle point.

(c) (25 pts) Suppose that $f(x,y,z)$ and $g(x,y,z)$ are scalar-valued functions with continuous first-order partial derivatives.
  (i) Show the product rule for the gradient

$$\nabla(fg) = f\nabla g + g\nabla f$$

  by calculating the components of the left hand side and the right side, showing that they are the same.
  (ii) Now, suppose that $f(x,y,z) = e^{x^2+y^2+z^2}$ and $g(x,y,z) = \arctan\left(\frac{xz+y}{4\pi^2}\right)$. Evaluate the line integral

$$\int_C \mathbf{F}\cdot d\mathbf{r}$$

  for the work done by the force field $\mathbf{F} = f\nabla g + g\nabla f$ on a particle that makes one revolution around the 'conical helix' given by

$$\mathbf{r}(t) = \langle t\cos(t), t\sin(t), t\rangle, \qquad 0 \le t \le 2\pi.$$

  Simplify your answer. *Hint: What property of* $\mathbf{F}$ *can help you with this problem?*

(d) (15 pts) Let $f(x,y)$ be a continuous function that has continuous partial derivatives. Suppose that $f_y(1,3) = 3$ and $D_{\mathbf{u}}f(1,3) = -\sqrt{5}$ where $\mathbf{u} = \langle -2, 1\rangle/\sqrt{5}$. Find the directional derivative of f at $(1,3)$ in the direction $\mathbf{v} = \langle 2, -4\rangle/\sqrt{20}$.

<u>**SOLUTION:**</u>

(a) (i) The velocity vector is $\mathbf{v}(t) = \mathbf{r}'(t) = \langle 0, -2\sin 2t, 2\sin 2t\rangle$; the acceleration vector is $\mathbf{a}(t) = \mathbf{r}''(t) = \langle 0, -2\cos 2t, 2\cos 2t\rangle = \langle 0, -2\cos^2 t + 2\sin^2 t, 2\cos^2 t - 2\sin^2 t\rangle$.

(ii) The arc length is the distance traveled, which depends on the particle speed: $|\mathbf{v}(t)| = |\mathbf{r}'(t)| = |\langle 0, -\sin 2t, \sin 2t\rangle| = \sqrt{2\sin^2 2t} = \sqrt{2}|\sin 2t|$. To compute the total distance traveled L, we need to integrate the speed in time

$$\begin{aligned} L &= \int_0^\pi \left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right| \,\mathrm{d}t \\ &= \int_0^\pi \sqrt{2}|\sin 2t|\,\mathrm{d}t \\ &= \int_0^{\pi/2} \sqrt{2}\sin 2t\,\mathrm{d}t + \int_{\pi/2}^{\pi} \sqrt{2}(-\sin 2t)\,\mathrm{d}t \\ &= -\frac{\sqrt{2}}{2}\cos 2t\Big|_{t=0}^{\pi/2} + \frac{\sqrt{2}}{2}\cos 2t\Big|_{t=\pi/2}^{\pi} \\ &= \sqrt{2}+\sqrt{2} = 2\sqrt{2} \qquad . \end{aligned}$$

(b) (i)

$$\begin{aligned} f_x &= 2x - 2xy = 0 \\ &\implies 2x(1-y) = 0 \implies x = 0 \text{ or } y = 1 \\ f_y &= 4y - x^2 = 0 \\ &\implies 4y = x^2 \end{aligned}$$

If $x = 0$, then the second equation requires $y = 0$. If $y = 1$, the second equation requires $x = \pm 2$. Thus the critical points are $(0,0)$, $(2,1)$ and $(-2,1)$.

(ii)

(x,y)	$f_{xx} = 2-2y$	$f_{yy} = 4$	$f_{xy} = -2x$	$D = 8-8y-4x^2$	Type
$(0,0)$	2	4	0	8	local minimum
$(2,1)$	0	4	-4	-16	saddle point
$(-2,1)$	0	4	4	-16	saddle point

(c) (i)

$$\begin{aligned} \nabla(fg) &= \langle (fg)_x, (fg)_y, (fg)_z\rangle \\ &= \langle fg_x + f_xg, fg_y + f_yg, fg_z + f_zg\rangle \quad \text{product rule} \\ &= \langle fg_x, fg_y, fg_z\rangle + \langle f_xg, f_yg, f_zg\rangle \\ &= f\langle g_x, g_y, g_z\rangle + g\langle f_x, f_y, f_z\rangle \\ &= f\nabla g + g\nabla f \end{aligned}$$

(ii) The field $\mathbf{F}$ is conservative since by part (i), $\mathbf{F} = \nabla(fg)$. That is, $\varphi(x,y,z) = f(x,y,z)g(x,y,z)$ is a potential function for $\mathbf{F}$. By the Fundamental Theorem of Line Integrals,

$$\begin{aligned} \int_C \mathbf{F}\cdot d\mathbf{r} &= \varphi(\mathbf{r}(2\pi)) - \varphi(\mathbf{r}(0)) \\ &= \varphi(2\pi,0,2\pi) - \varphi(0,0,0) \\ &= f(2\pi,0,2\pi)g(2\pi,0,2\pi) - f(0,0,0)g(0,0,0) \\ &= e^{4\pi^2+0+4\pi^2}\arctan\left(\frac{4\pi^2+0}{4\pi^2}\right) - e^0\underbrace{\arctan(0)}_{=0} \\ &= e^{8\pi^2}\cdot\frac{\pi}{4} \end{aligned}$$

(d) We want to find $D_{\mathbf{v}}f(1,3)$ where $\mathbf{v} = \langle \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\rangle$. We're given $f_y(1,3) = 3$, which implies $\nabla f(1,3) = \langle f_x(1,3), 3\rangle$. We're also given $D_{\mathbf{u}}f(1,3) = -\sqrt{5}$ where $\mathbf{u} = \langle \frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\rangle$, which implies

$\langle f_x(1,3), 3\rangle \cdot \langle \frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\rangle = -\sqrt{5}$. This implies

$$\begin{aligned} -2f_x(1,3) + 3 = -5 &\implies f_x(1,3) = 4 \\ &\implies \nabla f(1,3) = \langle 4, 3\rangle \\ &\implies D_{\mathbf{v}} f(1,3) = \langle 4,3\rangle \cdot \langle \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\rangle = -\frac{2}{\sqrt{5}} \end{aligned}$$

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Problem 3: (55 points)
You are enjoying an ice cream cone with your friend whose cell phone is emitting electromagnetic energy by the vector electric field $\mathbf{E} = \frac{x^3}{3}\,\mathbf{i} + \frac{y^3}{3}\,\mathbf{j} + \frac{z^3}{3}\,\mathbf{k}$. During the course of the conversation your friend asks you to calculate the outward flux of this vector field through your ice cream cone. The cone itself is the surface $z = \sqrt{x^2+y^2}, z \leq 1$ and the ice cream is contained in the portion of the sphere $z = 1 + \sqrt{1 - x^2 - y^2}$ with $z \geq 1$. Even though your friend's phone is emitting a lot of energy, it is not enough to make you want to do two surface integrals. Instead, since your ice cream cone and the vector field satisfy the hypotheses of the Divergence Theorem you decide to use that to find the flux.

(a) Set up, but DO NOT EVALUATE, the appropriate integral in Cartesian coordinates $(\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x)$ that uses the Divergence Theorem to find the required flux.
(b) The computation in part (a) appears rather difficult so you decide to try spherical coordinates with the hope that you can actually do the computation. Set up the integral using these coordinates and the order $\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$. DO NOT EVALUATE...YET.
(c) Things should be looking pretty good about now. Evaluate your spherical coordinates integral to find the flux.

SOLUTION:

(a) The flux F is

$$F = \iint_{\mathcal{S}} \mathbf{E}\cdot \mathrm{d}\mathbf{S} = \iiint_{\mathcal{W}} \nabla\cdot\mathbf{E}\,\mathrm{d}V, \qquad .$$

Begin by finding the divergence of $\mathbf{E}$ as

$$\nabla\cdot\mathbf{E} = x^2 + y^2 + z^2.$$

The cone and the sphere intersect where their z values are both 1, giving the curve of intersection as $x^2 + y^2 = 1$. Thus

$$F = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{1-\sqrt{1-x^2-y^2}} \left(x^2+y^2+z^2\right)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

(b) The cone's equation in spherical coordinates:

$$z = r \implies \rho\cos\phi = \rho\sin\phi \implies \tan\phi = 1 \implies \phi = \frac{\pi}{4}$$

The sphere's equation in spherical coordinates:

$$\begin{aligned} \rho\cos\phi &= 1 + \sqrt{1-\rho^2\sin^2\phi} \\ \rho^2\cos^2\phi - 2\rho\cos\phi + 1 &= 1 - \rho^2\sin^2\phi \\ \rho^2\cos^2\phi + \rho^2\sin^2\phi &= 2\rho\cos\phi \\ \rho &= 2\cos\phi \end{aligned}$$

The integrand is simply ρ^2. Thus

$$\iint_{\mathcal{S}} \mathbf{E}\cdot\mathrm{d}\mathbf{S} = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{2\cos\phi} \rho^4 \sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$$

(c)

$$
\begin{aligned}
\iint_{\mathcal{S}} \mathbf{E} \cdot \mathrm{d}\mathbf{S} &= \int_0^{2\pi} \int_0^{\pi/4} \int_0^{2\cos\phi} \rho^4 \sin\phi \, \mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}\theta \\
&= \int_0^{2\pi} \int_0^{\pi/4} \frac{1}{5}\rho^5 \bigg|_0^{2\cos\phi} \sin\phi \, \mathrm{d}\phi \, \mathrm{d}\theta \\
&= \frac{32}{5} \int_0^{2\pi} \int_0^{\pi/4} \cos^5\phi \sin\phi \, \mathrm{d}\phi \, \mathrm{d}\theta \qquad \begin{array}{ll} u = \cos\phi & \mathrm{d}u = -\sin\phi \, \mathrm{d}\phi \\ \phi = 0 \Rightarrow u = 1 & \phi = \pi/4 \Rightarrow u = 1/\sqrt{2} \end{array} \\
&= -\frac{32}{5} \int_0^{2\pi} \int_1^{1/\sqrt{2}} u^5 \, \mathrm{d}u \, \mathrm{d}\theta \\
&= -\frac{16}{15} \int_0^{2\pi} u^6 \bigg|_1^{1/\sqrt{2}} \mathrm{d}\theta \\
&= -\frac{16}{15}\left(-\frac{7}{8}\right) \int_0^{2\pi} \mathrm{d}\theta = \frac{28}{15}\pi
\end{aligned}
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Problem 4: (55 points)
Let C be the curve of intersection of the ellipsoid $4x^2 + y^2 + z^2 = 25$ with the plane $x = -2$, where C is traversed in the counterclockwise direction as viewed from the origin.

(a) Parameterize C. Be sure to give bounds for your parameter.
(b) Determine $\mathbf{T}$ and $\mathbf{B}$ for the curve C at the point $(-2, \frac{3}{2}, \frac{3\sqrt{3}}{2})$.
(c) Determine the curvature κ for the curve C at the point $(-2, \frac{3}{2}, \frac{3\sqrt{3}}{2})$.
(d) Let $\mathbf{F} = (xy - x)\mathbf{i} + xz\mathbf{j} + x^2y\mathbf{k}$. Find the circulation of $\mathbf{F}$ around C.

<u>**SOLUTION:**</u>

(a) Answers may vary. One possibility $\mathbf{r}(t) = \langle -2, 3\cos t, 3\sin t\rangle$, $0 \le t \le 2\pi$.

(b) $\mathbf{T}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \frac{\langle 0, -3\sin t, 3\cos t\rangle}{3} = \langle 0, -\sin t, \cos t\rangle$.
At the point $(-2, \frac{3}{2}, \frac{3\sqrt{3}}{2})$, $t = \pi/3$. Thus $\mathbf{T}(\pi/3) = \langle 0, -\frac{\sqrt{3}}{2}, \frac{1}{2}\rangle$
Since the curve lies in the plane $x = -2$, oriented counterclockwise, $\mathbf{B} = \langle 1, 0, 0\rangle$

(c) Curvature of a circle of radius a is $\frac{1}{a}$ thus $\kappa = \frac{1}{3}$ (or directly calculate $\frac{|\frac{dT}{dt}|}{|\frac{ds}{dt}|} = \frac{1}{3}$)

(d) Calculate the line integral directly or use Stokes' Theorem to convert it to a surface integral.
- Option 1: Line integral: $\frac{d}{dt}\mathbf{r}(t) = \langle 0, -3\sin t, 3\cos t\rangle$.

So we get

$$\begin{aligned}
\oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} &= \int_0^{2\pi} \langle -6\cos t + 2, -6\sin t, 12\cos t\rangle \cdot \langle 0, -3\sin t, 3\cos t\rangle \,\mathrm{d}t \\
&= \int_0^{2\pi} (18\sin^2 t + 36\cos^2 t)\,\mathrm{d}t \\
&= \int_0^{2\pi} (18 + 18\cos^2 t)\,\mathrm{d}t \\
&= 18\int_0^{2\pi} (1 + \frac{1+\cos(2t)}{2}))\,\mathrm{d}t \\
&= 18\left[\frac{3}{2}t + \frac{\sin(2t)}{4}\right]_0^{2\pi} = 54\pi
\end{aligned}$$

- Option 2: Stokes': $\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,\mathrm{d}\sigma$
  Easiest S to choose is the circle in the plane $x = -2$ of radius 3 centered at $(-2, 0, 0)$. Thus $\mathbf{n} = \langle 1, 0, 0\rangle$ and $\nabla\times\mathbf{F} = \langle x^2 - x, -2xy, z - y\rangle$
  Thus

$$\begin{aligned}
\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,\mathrm{d}\sigma &= \iint_S \langle x^2 - x, -2xy, z - y\rangle \cdot \langle 1, 0, 0\rangle\,\mathrm{d}\sigma \\
&\iint_S (x^2 - x)\,\mathrm{d}\sigma \\
&= \iint_S ((-2)^2 - (-2))\,\mathrm{d}\sigma \\
&= 6\iint_S \mathrm{d}\sigma \\
&= 6(\text{ Area of S}) \\
&= 6\pi(3)^2 \\
&= 54\pi
\end{aligned}$$

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Problem 5: (55 points)
Consider the quadric surface given by $x^2 + y^2 - z^2 = 1$.

(a) What is the name of this quadric surface?
(b) Suppose that a thin metal sheet is bent into the shape of the portion of the quadric surface S for $0 \le z \le 2$ and that its density at any point is four times the z-coordinate of that point. Find the mass of the thin metal sheet. Simplify your answer.
(c) Solve the quadric surface equation for $z = f(x, y)$, a function of x and y, when $z \ge 0$. Find the linearization of $f(x, y)$ at the point $(2, 0)$ and use it to approximate $f(2.2, 0.1)$.

<u>SOLUTION:</u>

(a) Hyperboloid of one sheet.
(b) The density function is $\rho(x, y, z) = 4z$ and the mass is

$$M = \iint_S \rho(x, y, z)\,\mathrm{d}\sigma.$$

To evaluate this surface integral, we convert to a double integral. Set $g(x, y, z) = x^2 + y^2 - z^2$ so that the surface is the level surface $g(x, y, z) = 1$.
We will project onto the x-y plane eventually, so $\mathbf{p} = \mathbf{k}$ and everything must be written in terms of

x and y only. Note that $\nabla g = \langle 2x, 2y, -2z\rangle$ so that

$$\begin{aligned}|\nabla g| &= \sqrt{4x^2+4y^2+4z^2}\\ &= \sqrt{4x^2+4x^2+4(x^2+y^2-1)}\\ &= 2\sqrt{2x^2+2y^2-1}\end{aligned}$$

and

$$|\nabla g\cdot \mathbf{p}| = |\langle 2x, 2y, -2z\rangle\cdot\langle 0,0,1\rangle| = |-2z| = 2z = 2\sqrt{x^2+y^2-1}$$

since $z \geq 0$ on S.

The projection of S onto the xy-plane is an annulus R with inner radius 1 and outer radius $\sqrt{5}$. Thus

$$\begin{aligned}M &= \iint_S 4z\,\mathrm{d}\sigma\\ &= \iint_R 4\sqrt{x^2+y^2-1}\underbrace{\left(\frac{2\sqrt{2x^2+2y^2-1}}{2\sqrt{x^2+y^2-1}}\right)}_{\frac{|\nabla g|}{|\nabla g\cdot\mathbf{p}|}}\mathrm{d}A\\ &= 4\iint_R \sqrt{2x^2+2y^2-1}\,\mathrm{d}A\end{aligned}$$

Because of the integrand and the region R we convert to polar coordinates to obtain

$$\begin{aligned}M &= 4\int_0^{2\pi}\int_1^{\sqrt{5}}\sqrt{2r^2-1}\,r\,\mathrm{d}r\,\mathrm{d}\theta\\ &= 4\cdot 2\pi\int_1^9\sqrt{u}\,\frac{\mathrm{d}u}{4} \qquad u = 2r^2-1 \implies \mathrm{d}u = 4r\,\mathrm{d}r\\ &= 2\pi\left[\frac{2}{3}u^{3/2}\right]_1^9\\ &= \frac{4\pi}{3}\left(9^{3/2}-1^{2/3}\right)\\ &= \frac{4\pi}{3}(27-1)\\ &= \frac{104\pi}{3}.\end{aligned}$$

(c) From part (b) we know that $z = \sqrt{x^2+y^2-1}$ for $z \geq 0$, so that $f(x,y) = \sqrt{x^2+y^2-1}$. The linearization is

$$\begin{aligned}L(x,y) &= f(2,0)+f_x(2,0)(x-2)+f_y(2,0)y\\ &= \sqrt{3}+\frac{2}{\sqrt{3}}(x-2)\\ &= \frac{2}{\sqrt{3}}x-\frac{1}{\sqrt{3}} = \frac{2\sqrt{3}}{3}x-\frac{\sqrt{3}}{3}.\end{aligned}$$

Then

$$f(2.2,0.1)\approx L(2.2,0.1) = \frac{1}{\sqrt{3}}(2(2.2)-1) = \frac{3.4}{\sqrt{3}} = \frac{3.4\sqrt{3}}{3}$$

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