APPM 2350—Section Exam 3—140 points

Wednesday November 28, 6:00pm – 7:30pm, 2018

ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your student ID number, (3) lecture section/time (4) your instructor's name, and (5) a grading table. Text books, class notes, and calculators are NOT permitted. A one-page one-sided crib sheet is allowed.

Problem 1 – True/False: (20 points)
For the following true/false questions, write TRUE (for always true) or FALSE (if not always true). Your work will not be graded.

(a) The force field $\mathbf{F}(x, y) = \langle -16 - 5y - 3x^2, -7 - 5x - 4y^2 \rangle$ acts mostly against the movement of a particle that travels once counterclockwise around the triangle with corners $(1, 0)$, $(0, 1)$, $(-1, 0)$ because the work done is negative.

(b) For any function $f(x, y)$ continuous on all of $\mathbb{R}^2$, $\displaystyle\int_0^1 \int_0^x f(x, y)\, dy\, dx = \int_0^1 \int_0^y f(x, y)\, dx\, dy.$

(c) A line parallel to the y-axis through the center of mass of a closed, connected, two-dimensional lamina in the x-y plane divides the lamina into subregions of equal area.

(d) Suppose the curve C is the straight line path from the origin to the point (π, π), then the line integral $\int_C f(x, y)\, \mathrm{d}s$ can be written $\int_0^\pi f(t, t)\, \mathrm{d}t$.

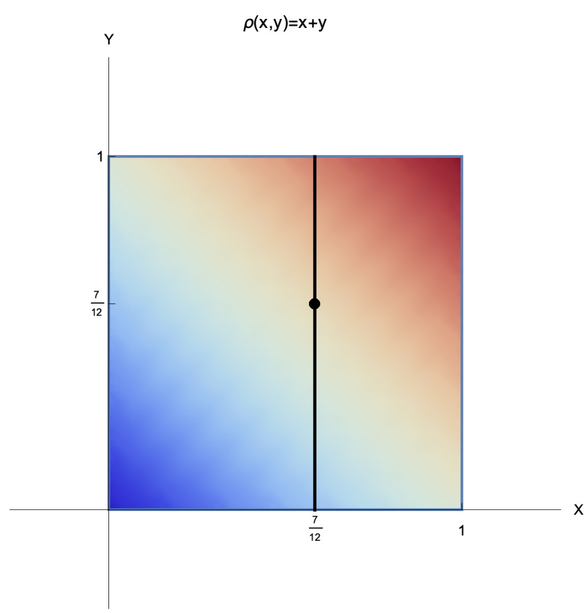
SOLUTION:

(a) FALSE: The field is conservative since if we set $\mathbf{F} = \langle P, Q \rangle$, then $\frac{\partial Q}{\partial x} = -5 = \frac{\partial P}{\partial y}$ and the domain of $\mathbf{F}$, which is all of $\mathbb{R}^2$, is simply connected. So the work done is zero since the curve is closed.

(b) FALSE: the regions specified by both integrals aren't the same, so we expect it to be false. As a specific counterexample,

$$\int_0^1 \int_0^x x\, dy\, dx \neq \int_0^1 \int_0^y x\, dx\, dy$$

(c) FALSE: The density isn't necessarily constant. As a specific counterexample, consider the lamina in the shape of the square in $\mathbb{R}^2$ with corners $(0, 0), (1, 0), (1, 1), (0, 1)$ and with density function $\rho(x, y) = x + y$. The mass of the lamina is 1 and the the center of mass is $(7/12, 7/12)$. The line $x = 7/12$ divides the lamina into rectangles of area $7/12$ and $5/12$. See Figure below.



(d) False: the line can be parametrized as $\mathbf{r}(t) = \langle t, t \rangle$ for $0 \le t \le \pi$. Then

$$\int_C f(x,y)\,\mathrm{d}s = \int_0^\pi f(t,t)\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right|\,\mathrm{d}t = \int_0^\pi f(t,t)\sqrt{1+1}\,\mathrm{d}t.$$

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Problem 2 – Short Answer Questions: (40 points)
For the questions in this problem, show all your work and clearly $\boxed{\text{box}}$ your final answer. Partial credit may be given.

(a) The integral

$$I = \int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} \int_0^{\sqrt{4-x^2-y^2}} dz\,dx\,dy$$

calculates the volume of a 3D object in Cartesian coordinates. However, it's difficult to evaluate in these coordinates. Convert the integral I to an equivalent integral (or integrals) in spherical coordinates. (You DO NOT need to evaluate).

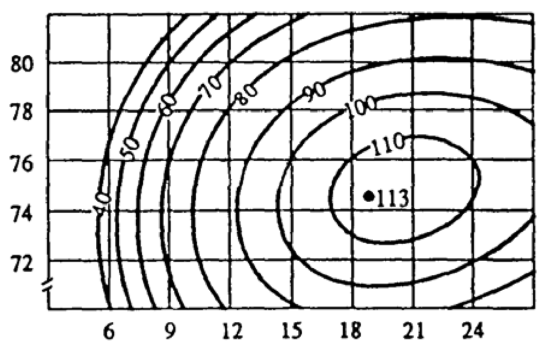

FIGURE 1. **Problems 2b, 2c, and 2d utilize this level curve (contour) graph of the function $g(x,y)$. In all these problems, the vector field $\mathbf{G}(x,y) = \nabla g(x,y)$ is used.**

(b) Let C be the level curve defined by $g(x,y) = 110$ and oriented counter-clockwise. Determine whether the work done by $\mathbf{G}$ around C is positive, zero, or negative. Justify your answer.

(c) Let C be the level curve defined by $g(x,y) = 110$ and oriented counter-clockwise. Determine whether the total flux of $\mathbf{G}$ out of C is positive, zero, or negative. Justify your answer.

(d) Let C' be the straight line path $\mathbf{r}(t)$ from $(x,y) = (18,76)$ to $(x,y) = (9,80)$. Evaluate the work integral $\int_{C'} \mathbf{G} \cdot \mathrm{d}\mathbf{r}$.

SOLUTION:

(a)

$$I = \int_0^{\pi/4} \int_0^{\pi/2} \int_0^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

(b) Since $\mathbf{G}$ is a conservative vector field, the work done by $\mathbf{G}$ around any closed curve is zero. Since $C : g(x, y) = 110$ is a closed curve, the work done is zero.

(c) The total flux is $\oint_C \mathbf{G} \cdot \mathbf{n} \, \mathrm{d}s$ where $\mathbf{n}$ is the outward pointing unit normal. Recall that $\mathbf{n} = -\nabla g / |\nabla g|$ is the unit normal to any level curve of $g(x, y)$. Then the total flux is

$$\oint_C \mathbf{G} \cdot \mathbf{n} \, \mathrm{d}s = \oint_C \nabla g \cdot \left(-\frac{\nabla g}{|\nabla g|} \right) \mathrm{d}s$$
$$= -\oint_C |\nabla g| \, \mathrm{d}s < 0$$

-or-

The outward normal points in the direction of decreasing g so the flux is negative.

(d) $\int_C \nabla g \cdot d\mathbf{r} = g(9, 80) - g(18, 76) = 50 - 110 = -60$

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Problem 3: (40 points)
On a certain day, the density of lightning strikes in the Colorado Rocky Mountains was found to be well-approximated by the function

$$\delta(x, y) = \left(x^2 + y^2\right)^{3/2} \left[1 + 12 \tan^{-1}(y/x)\right]$$

strikes per unit area. Some colleagues of yours need to calculate the total number of lightning strikes N that day in a certain area to help determine the risk of new forest fires. They have decided that this can be accomplished by computing the value of

$$N = \int_{\sqrt{3}/2}^{1} \int_{\sqrt{1-x^2}}^{x/\sqrt{3}} \delta(x, y) \, \mathrm{d}y \, \mathrm{d}x + \int_1^{\sqrt{3}} \int_0^{x/\sqrt{3}} \delta(x, y) \, \mathrm{d}y \, \mathrm{d}x + \int_{\sqrt{3}}^{2} \int_0^{\sqrt{4-x^2}} \delta(x, y) \, \mathrm{d}y \, \mathrm{d}x.$$

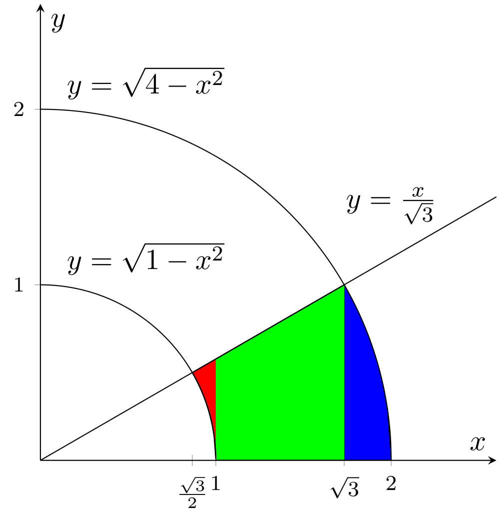
However, since they took Calculus 3 many moons ago, they cannot recall how to do this calculation. Come to their assistance by determining the total number N of lightning strikes in their area of interest, simplifying your final answer. *Hint: draw the region of integration. Another coordinate system will be useful.*

SOLUTION:

The form of the density function strongly suggests the use of polar coordinates, giving

$$\delta(r, \theta) = r^3 (1 + 12\,\theta)$$

Using the bounds on the integrals, we can draw the following figure, which also suggests using polar coordinates:



From the figure:

$$y = \sqrt{1 - x^2} \implies x^2 + y^2 = 1 \implies r = 1$$
$$y = \sqrt{4 - x^2} \implies x^2 + y^2 = 4 \implies r = 2$$
$$y = x/\sqrt{3} \implies y/x = 1/\sqrt{3} \implies \tan^{-1}(y/x) = \tan^{-1}(1/\sqrt{3}) \implies \theta = \pi/6$$

so the bounds of integration in polar coordinates are $1 \leq r \leq 2$ and $0 \leq \theta \leq \pi/6$. Thus the total number of lightning strikes becomes

$$\int_0^{\pi/6} \int_1^2 r^3(1 + 12\theta) r \, \mathrm{d}r \, \mathrm{d}\theta = \int_0^{\pi/6} \int_1^2 r^4(1 + 12\theta) \, \mathrm{d}r \, \mathrm{d}\theta$$
$$= \int_0^{\pi/6} (1 + 12\theta) \frac{1}{5} r^5 \Big|_1^2 \mathrm{d}\theta = \frac{31}{5} \int_0^{\pi/6} (1 + 12\theta) \, \mathrm{d}\theta = \frac{31}{5} \left(\theta + 6\theta^2 \right) \Big|_0^{\pi/6}$$
$$= \frac{31}{5} \left[\frac{\pi}{6} + 6 \left(\frac{\pi^2}{36} \right) \right] = \frac{31\pi}{30}(1 + \pi) \text{ lightning strikes.}$$

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Problem 4: (40 points)
Your mechanical engineering friend designs a new part using CAD software. An important component of the part is determined by the curve C that is given by the intersection of the surfaces $y = x^2 + 9$ and $z = 3$ in three dimensions.

(a) A wire is placed along the curve C from $(2, 13, 3)$ to $(3, 18, 3)$ whose density at a point on the wire is given by its x-coordinate. Unfortunately, the CAD software is not capable of determining the mass of the wire so, your friend needs your help. Find the mass of the wire.
(b) Let $\mathbf{G}(x, y, z) = 2xy\mathbf{i} + (x^2 + 2z)\mathbf{j} + 2y\mathbf{k}$ be a vector force field. Show that $\mathbf{G}$ is conservative and find its potential function $g(x, y, z)$ so that $\mathbf{G}(x, y, z) = \nabla g(x, y, z)$.
(c) Find the work done on the particle by the force field $\mathbf{G}$ when moving along C from $(0, 9, 3)$ to $(2, 13, 3)$.

SOLUTION:

The curve C can be parametrized by its x coordinate as $\mathbf{r}(t) = \langle t, t^2 + 9, 3 \rangle$ where t varies appropriately for each part of the problem.

(a) The density of the wire is $\delta(x,y,z) = x$ per unit length ; the portion of the curve C considered here can be parametrized as $\mathbf{r}(t)$ with $2 \le t \le 3$; its mass is then the line integral

$$M = \int_C \delta(x,y,z)\,\mathrm{d}s = \int_{t=2}^{3} t\sqrt{1+4t^2}\,\mathrm{d}t$$

$$\text{let } u = 4t^2 + 1 \;\Rightarrow\; \mathrm{d}u = 8t\,\mathrm{d}t,$$

$$= \frac{1}{8}\int_{u=17}^{37} \sqrt{u}\,\mathrm{d}u$$

$$= \frac{1}{12}u^{3/2}\Big|_{u=17}^{37} = \frac{1}{12}(37^{3/2} - 17^{3/2}) \quad .$$

(b) Let $P = 2xy$, $Q = x^2 + 2z$, $R = 2y$ be the components of the vector field $\mathbf{G}$. We verify the test for conservatism

$$P_y = 2x = Q_x, \quad P_z = 0 = R_x, \quad Q_z = 2 = R_y.$$

To obtain the potential g, we compute

$$g_x = P = 2xy \quad \Rightarrow \quad g(x,y,z) = x^2y + f(y,z)$$

$$\Rightarrow g_y = x^2 + f_y = Q = x^2 + 2z \quad \Rightarrow \quad f(y,z) = 2yz + h(z)$$

$$\Rightarrow g_z = 2y + h'(z) = R = 2y \quad \Rightarrow h(z) = const$$

$$\Rightarrow g(x,y,z) = x^2y + 2yz.$$

(c) The work done W on the particle by $\mathbf{G}$ can now be evaluated using the fundamental theorem of line integration

$$W = \int_C \mathbf{G}\cdot\mathrm{d}\mathbf{r}$$

$$= g(2,13,3) - g(0,9,3)$$

$$= 2^2 \cdot 13 + 2\cdot 13\cdot 3 - (0^2\cdot 9 + 2\cdot 9\cdot 3) = 76.$$

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