APPM 2350—Section Exam 2—140 points
Wednesday October 24, 6:00pm – 7:30pm, 2018

ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your student ID number, (3) lecture section/time (4) your instructor's name, and (5) a grading table. Text books, class notes, and calculators are NOT permitted. A one-page one-sided crib sheet is allowed.

Problem 1 – True/False: (20 points)
For the following true/false questions, write TRUE (for always true) or FALSE (if not always true). Your work will not be graded.

(a) If $f(x,y) \to L$ as $(x,y) \to (a,b)$ along *every straight line* passing through (a,b), then

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

(b) Given $f(x(r,s), y(r), z(r,s,t))$ then

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{dy}{dr} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r}$$

(c) Given $f(x,y,z) = xyz$ the smallest (ie most negative) derivative of $f(x,y,z)$ at the point $(1,2,3)$ (out of all possible directions) is -7.

(d) The vector $\nabla f(a,b)$ is normal to the graph of $z = f(x,y)$ at the point $(a,b,f(a,b))$.

<u>**SOLUTION:**</u>

(a) False. In order for the limit to exist, the limit along *every* path through that point must agree, not just the straight line paths.

(b) False. $\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{dy}{dr} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r}$

(c) True. The smallest derivative is $-|\nabla f(1,2,3)|$.
$\nabla f(x,y,z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$
$\implies \nabla f(1,2,3) = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k} \implies -|\nabla f(1,2,3)| = -\sqrt{6^2 + 2^2 + 3^2} = -7$

(d) False. The vector $\nabla f(x,y)$ is normal to the level curves of $f(x,y)$. Thus the vector $\nabla f(x,y)$ is normal to the graph of the *level curve* $f(x,y) = f(a,b)$ at the point $(x,y) = (a,b)$.

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Problem 2 – Short Answer Questions: (60 points)
For the questions in this problem, show all your work and clearly $\boxed{\text{box}}$ your final answer. Partial credit may be given.

(a) Find $\partial z/\partial y$ for $z = z(x,y)$ that satisfies the relation $\ln(xz) + \sin(yz) - \frac{z^2}{2} = \pi$.

(b) Find a value of the constant c that makes the following function continuous at $(0,0)$:

$$f(x,y) = \begin{cases} \dfrac{e^{x^2+y^2} - 1}{3(x^2+y^2)} & \text{if } (x,y) \neq (0,0) \\ c & \text{if } (x,y) = (0,0) \end{cases}$$

(c) If $g(s,t) = f(x(s,t), y(s,t))$ where $x(s,t) = s+t$ and $y(s,t) = s-t$, find a simplified expression for g_{tt} in terms of derivatives of f. Assume all partial derivatives involved exist and are continuous.

(d) At a certain point P on a mountain, a skier looks directly east and measures a 30° drop-off (decline), then looks directly north and measures a 45° rise (incline). Using these two measurements of the mountain topography, find a vector that points in the direction of steepest **descent** from the point P.

<u>**SOLUTION:**</u>

(a) We use implicit differentiation

$$\frac{\partial}{\partial y}\left(\ln(xz) + \sin(yz) - \frac{z^2}{2}\right) = \frac{\partial}{\partial y}(\pi)$$
$$\Downarrow$$
$$\frac{1}{xz}x\frac{\partial z}{\partial y} + \cos(yz)\left(z + y\frac{\partial z}{\partial y}\right) - z\frac{\partial z}{\partial y} = 0$$
$$\Downarrow$$
$$\frac{\partial z}{\partial y}\left(\frac{1}{z} + y\cos(yz) - z\right) = -z\cos(yz)$$
$$\Downarrow$$
$$\boxed{\frac{\partial z}{\partial y} = -\frac{z\cos(yz)}{\frac{1}{z} + y\cos(yz) - z}}$$

or

$$\boxed{\frac{\partial z}{\partial y} = -\frac{z^2\cos(yz)}{1 + yz\cos(yz) - z^2}}$$

(b) Converting to polar coordinates, note that

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} f(x,y) &= \lim_{r\to 0^+} \frac{e^{r^2} - 1}{3r^2} && \text{indeterminate } \frac{0}{0} \\
&= \lim_{r\to 0^+} \frac{2re^{r^2}}{6r} && \text{L'Hôpital's Rule} \\
&= \frac{1}{3}\lim_{r\to 0^+} e^{r^2} \\
&= \frac{1}{3},
\end{aligned}$$

so we should set $\boxed{c = \frac{1}{3}}$.

(c) Set $x = s + t$ and $y = s - t$. Then

$$\begin{aligned}
g_t &= f_x x_t + f_y y_t \\
&= f_x - f_y
\end{aligned}$$

and

$$\begin{aligned}
g_{tt} &= (f_x)_t - (f_y)_t \\
&= [(f_x)_x x_t + (f_x)_y y_t] - [(f_y)_x x_t + (f_y)_y y_t] \\
&= f_{xx} - f_{xy} - (f_{yx} - f_{yy}) \\
&= \boxed{f_{xx} - 2f_{xy} + f_{yy}}
\end{aligned}$$

where we have used Clairaut's theorem.

(d) If the elevation on the mountain range is given by $z = f(x,y)$, the given information says that

$$\frac{\partial f}{\partial x} = -\tan(30^\circ) = -\frac{1}{\sqrt{3}} \quad \text{and} \quad \frac{\partial f}{\partial y} = \tan(45^\circ) = 1,$$

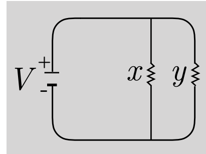

FIGURE 1. Two-resistor circuit with resistors x and y in parallel.

so the gradient at the point the skier measures is $\langle -\frac{1}{\sqrt{3}}, 1\rangle$. This points in the direction of steepest *ascent*, so the vector $\boxed{\langle \frac{1}{\sqrt{3}}, -1\rangle}$ points in the direction of steepest *descent*.

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Problem 3: (30 points)
The total resistance R of two resistors that are connected in parallel, each with resistances x and y such as in the circuit in Fig. 1, are related according to

$$\frac{1}{R} = \frac{1}{x} + \frac{1}{y}.$$

(a) If x, y, and R are allowed to vary, find an expression for the differential dR in terms of the differentials dx and dy.
(b) You have designed a two-resistor circuit like the one in Fig. 1 to have resistances $x = 10$ Ohms and $y = 40$ Ohms, but there is always some variation in manufacturing and the resistors received by your firm will probably not have these exact values. Will the total resistance R be more sensitive to variation in x or to variation in y?
(c) Find the linear approximation (linearization) $L(x, y)$ of the function $R(x, y)$ about the resistances $(x, y) = (10, 40)$ Ohms.

SOLUTION:

(a) Differentiating the expression, we obtain

$$-dR\frac{1}{R^2} = -dx\frac{1}{x^2} - dy\frac{1}{y^2}.$$

Solving for dR,

$$dR = \left(\frac{R}{x}\right)^2 dx + \left(\frac{R}{y}\right)^2 dy.$$

(b) Inserting $x = 10$ and $y = 40$ into the result from part (a), we obtain

$$dR = \left(\frac{R}{10}\right)^2 dx + \left(\frac{R}{40}\right)^2 dy,$$

so that dR will be more sensitive to dx because $(R/10)^2 > (R/40)^2$.

(c) We need to compute the first partials R_x and R_y. Either from part (a) or direct calculation, we obtain

$$\frac{\partial R}{\partial x} = \frac{1}{\left(1 + \frac{x}{y}\right)^2}$$
$$\frac{\partial R}{\partial y} = \frac{1}{(1 + \frac{y}{x})^2}.$$

Evaluating at $(x, y) = (10, 40)$, we have

$$\begin{aligned} R(10,40) &= (1/10 + 1/40)^{-1} = (5/40)^{-1} = 8, \\ R_x(10,40) &= (1 + 1/4)^{-2} = (4/5)^2 = 16/25, \\ R_y(10,40) &= (1+4)^{-2} = 1/25. \end{aligned}$$

Then the linear approximation near $(x, y) = (10, 40)$ is

$$\begin{aligned} R(x,y) \approx L(x,y) &= 8 + \frac{16}{25}(x-10) + \frac{1}{25}(y-40) \\ &= \frac{16}{25}x + \frac{1}{25}y. \end{aligned}$$

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Problem 4: (30 points)
You recently moved to the spherical planet A2P3P5M0 whose radius you have been told is $4\sqrt{3}$ units. You need to find a place to live on the surface of the planet. Thankfully, before building a residence, your realtor told you about a subterranean radiation source located at $(1, -1, 1)$. Since you do not want to glow in the dark, you want to live as far from the radiation source as possible. Where on the planet surface should you build your house? You can assume the center of the planet is at the origin of a standard xyz–coordinate system. To obtain full credit, you must use Calculus III techniques that you have learned in this course.

SOLUTION:

Let (x, y, z) be the coordinates of your residence. We want to maximize the distance (squared for ease of calculation) between your residence and the radiation source, which we designate as

$$f(x,y,z) = (x-1)^2 + (y+1)^2 + (z-1)^2$$

Since we are required to live on the surface of the planet, the coordinates of our residence are constrained to satisfy

$$g(x,y,z) = x^2 + y^2 + z^2 = (4\sqrt{3})^2 = 48$$

Set up the Lagrange equations

$$\begin{array}{llll} f_x = 2(x-1) & g_x = 2x & f_x = \lambda g_x \implies 2(x-1) = 2x\lambda \\ f_y = 2(y+1) & g_y = 2y & f_y = \lambda g_y \implies 2(y+1) = 2y\lambda \\ f_z = 2(z-1) & g_z = 2z & f_z = \lambda g_z \implies 2(z-1) = 2z\lambda \end{array}$$

We need to solve the following system of simultaneous equations:

$$x - 1 = \lambda x \quad (1)$$
$$y + 1 = \lambda y \quad (2)$$
$$z - 1 = \lambda z \quad (3)$$
$$x^2 + y^2 + z^2 = 48 \quad (4)$$

Note that neither x nor y nor z can be zero since none of Eqs. (1)-(3) can be satisfied in that case. Hence we can write

$$\frac{x-1}{x} = \lambda = \frac{y+1}{y} \implies xy - y = xy + x \implies y = -x$$

and

$$\frac{x-1}{x} = \lambda = \frac{z-1}{z} \implies xz - z = xz - x \implies z = x$$

Eq. (4) then yields

$$x^2 + x^2 + x^2 = 3x^2 = 48 \implies x^2 = 16 \implies x = \pm 4$$

The critical points are $(4, -4, 4)$ and $(-4, 4, -4)$ and

$$f(4, -4, 4) = (4-1)^2 + (-4+1)^2 + (4-1)^2 = 27$$
$$f(-4, 4, -4) = (-4-1)^2 + (4+1)^2 + (-4-1)^2 = 75$$

Thus, we should place our residence at $(-4, 4, -4)$.

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