APPM 2350—Section Exam 1—140 points

Wednesday September 26, 6:00pm – 7:30pm, 2018

ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your student ID number, (3) lecture section/time (4) your instructor's name, and (5) a grading table. Text books, class notes, and calculators are NOT permitted. A one-page one-sided crib sheet is allowed.

Problem 0 – Multiple Choice: (1 point)
Which would you prefer?

(a) Office hours in ECCR 244 (can be crowded).
(b) Office hours in FLMG 208 (much larger space, less crowded, farther away from Engineering Center).

Problem 1 – True/False: (25 points)
For the following true/false questions, write TRUE (for always true) or FALSE (if not always true). Your work will not be graded.

(a) The unit tangent $\mathbf{T}$, unit normal $\mathbf{N}$, and unit binormal $\mathbf{B}$ vectors to a space curve $\mathbf{r}(t)$ satisfy the relationships $\mathbf{T} = \mathbf{N} \times \mathbf{B}$, $\mathbf{N} = \mathbf{B} \times \mathbf{T}$, $\mathbf{B} = \mathbf{T} \times \mathbf{N}$.
(b) An accelerated particle that is moving according to $\mathbf{r}(t) = \langle \cos t^2, \sin t^2 \rangle$, $0 \le t < \sqrt{2\pi}$ experiences constant curvature.
(c) A plane is determined by three distinct points that lie on it.
(d) A line in three dimensions is determined by two distinct points that lie on it.
(e) The shortest distance between a point P_1 and the line $\mathbf{r}(t)$ defined by the point P_0 and the parallel vector $\mathbf{v}$ with the parametric equation

$$\mathbf{r}(t) = \overrightarrow{OP_0} + t\mathbf{v}, \quad -\infty < t < \infty \quad \text{where} \quad O = (0,0,0)$$

is $|\text{proj}_{\mathbf{v}} \overrightarrow{P_0P_1}|$.

SOLUTION:

5 points each:

(a) TRUE (right hand rule)
(b) TRUE (curvature is $\kappa = 1$, no matter what parametrization of the circle is used)
(c) FALSE (if the three points are collinear, then they only determine a line)
(d) TRUE
(e) FALSE (the shortest distance is $\left|\text{orth}_{\mathbf{v}} \overrightarrow{P_0P_1}\right| = \left|\overrightarrow{P_0P_1} - \text{proj}_{\mathbf{v}} \overrightarrow{P_0P_1}\right|$.)

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Problem 2 – Short Answer Questions: (25 points)
For the questions in this problem, no motivation is required. Clearly $\boxed{\text{box}}$ your final answer. Only your boxed final answer will be graded.

(a) At a certain time t_0 a particle is moving with nonzero speed along a curve in $\mathbb{R}^3$ such that its velocity and acceleration vectors are scalar multiples of one another. At what rate is the particle's unit tangent vector changing with respect to arc length?
(b) Consider the collection of vectors of the form $a\mathbf{j} + b\mathbf{k}$ where a and b are real numbers. Find an equation of the plane in which the collection of vectors $\mathbf{i} \times [\mathbf{i} \times (a\,\mathbf{j} + b\,\mathbf{k})]$ lie.
(c) Describe the surface given by $x^2 - 6x + y^2 - 6y = z^2 + (y-2)^2 - 13$.
(d) Suppose the scalar triple product of three vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ is $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 0$. What conclusion can be drawn about the three vectors?

SOLUTION:

(a) (7 pts) $\boxed{0}$ Since $\mathbf{r}'$ and $\mathbf{r}''$ are scalar multiples of one another, their cross product vanishes. This implies that the curvature vanishes meaning that the unit tangent vector is not changing with respect to arc length.

$$\kappa = \left\| \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} \right\| = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = 0$$

(b) (6 pts) $\boxed{x = 0}$ $a\,\mathbf{j} + b\,\mathbf{k}$ lies in the yz–plane for all choices of a and b. The vectors in question are $-a\,\mathbf{j} - b\,\mathbf{k}$ and consequently also lie in the yz–plane, whose equation is $x = 0$.

$$\mathbf{i} \times [\mathbf{i} \times (a\,\mathbf{j} + b\,\mathbf{k})] = \mathbf{i} \times (a\,\mathbf{k} - b\,\mathbf{j}) = -a\,\mathbf{j} - b\,\mathbf{k}$$

(c) (6 pts) $\boxed{\text{hyperbolic paraboloid}}$ After completing the square in x and performing some algebraic manipulations the given polynomial can be written as

$$y = \frac{1}{2}(x-3)^2 - \frac{1}{2}z^2$$

(d) (6 pts) $\boxed{\text{The vectors are coplanar}}$ The absolute value of the scalar triple product is the volume of the parallelepiped formed by the three vectors. Since this volume is zero, there is no parallelepiped, implying that the vectors all lie in the same plane. Viewed another way, recall that the cross product of two vectors is orthogonal to both of them. These two vectors lie in a plane. If the dot product of this cross product and the third vector vanishes, then the third vector is orthogonal to the cross product of the original two vectors, meaning that the third vector must lie in the same plane as the first two.

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SEE PROBLEMS ON THE OTHER SIDE

Problem 3: (30 points)

(a) Show that if the vectors $\mathbf{u}$ and $\mathbf{v}$ have the same length, then $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$ must be orthogonal.
(b) You might remember the "law of cosines," which says that

$$c^2 = a^2 + b^2 - 2ab\cos\theta,$$

where a, b and c are the lengths of the sides of a (not necessarily right) triangle and θ is the angle between the sides of lengths a and b. Use vectors and any properties of vectors learned in class to prove the law of cosines.

(c) Consider the curve given by $\mathbf{r}(t) = \langle t^2+1, t, 0\rangle$ for $t \geq 0$.
(i) Find the unit tangent vectors $\mathbf{T}(t)$, $t \geq 0$ and, in particular, $\mathbf{T}(1)$.
(ii) Sketch a graph of the parametric curve in the plane $z = 0$.
(iii) **Without calculating** $\mathbf{N}(t)$ (seriously, don't do it), find the unit normal $\mathbf{N}(1)$.
(iv) Find the unit binormal $\mathbf{B}(1)$.

SOLUTION:

(a) **(10 pts)** Assume $|\mathbf{u}| = |\mathbf{v}|$. Then

$$\begin{aligned}(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) &= (\mathbf{u}+\mathbf{v})\cdot\mathbf{u} - (\mathbf{u}+\mathbf{v})\cdot\mathbf{v}\\ &= \mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{v}\\ &= |\mathbf{u}|^2 - |\mathbf{v}|^2\\ &= 0\end{aligned}$$

since $\mathbf{u}$ and $\mathbf{v}$ have the same length. Thus $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$ are orthogonal.
(b) **(10 pts)** Let $\mathbf{a}$ be a vector of length a, $\mathbf{b}$ be a vector of length b, and assume they form two sides of a triangle where θ is the angle between them. Then $\mathbf{a}-\mathbf{b}$ forms the third side, and note that

$$\begin{aligned}c^2 &= |\mathbf{a}-\mathbf{b}|^2\\ &= (\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})\\ &= |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\mathbf{a}\cdot\mathbf{b}\\ &= |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos\theta\\ &= a^2 + b^2 - 2ab\cos\theta\end{aligned}$$

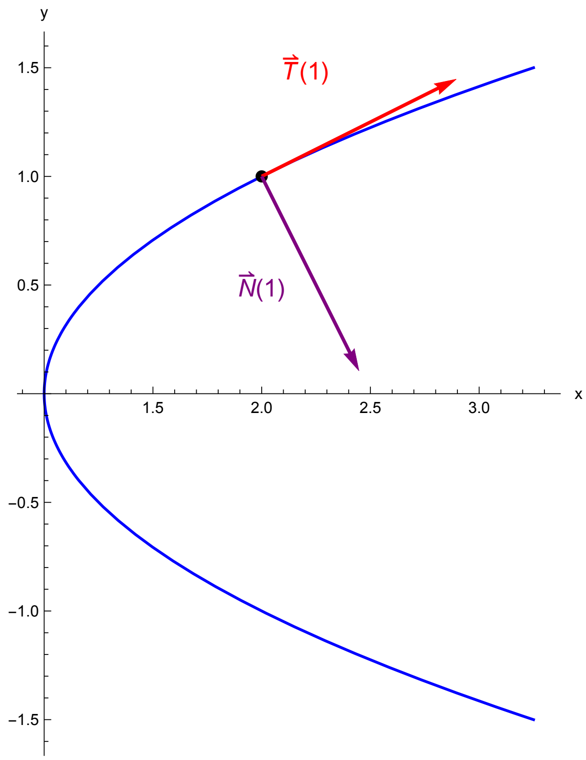
(c)
(i) **(3 pts)**

$$\mathbf{r}'(t) = \langle 2t, 1\rangle \implies |\mathbf{r}'(t)| = \sqrt{4t^2+1} \implies \mathbf{T}(t) = \left\langle \frac{2t}{\sqrt{4t^2+1}}, \frac{1}{\sqrt{4t^2+1}}\right\rangle.$$

Also,

$$\mathbf{T}(1) = \langle 2/\sqrt{5}, 1/\sqrt{5}\rangle.$$

(ii) **(2 pts)** It's $x = y^2 + 1$ and looks like this:



(iii) **(3 pts)** Since $\mathbf{N}(1)$ must be orthogonal to $\mathbf{T}(1)$, there are two candidates you can guess by inspection, $\langle 1/\sqrt{5}, -2/\sqrt{5}\rangle$ or $\langle -1/\sqrt{5}, 2/\sqrt{5}\rangle$. Looking at the curve at the point $(2,1)$, the unit normal must point rightwards and downwards as it points into the "concave side of the curve." Thus $\mathbf{N}(1) = \langle 1/\sqrt{5}, -2/\sqrt{5}\rangle$.

(iv) **(2 pts)** Because this is a curve in the xy-plane, we must have $\mathbf{B}(t) = \pm\mathbf{k}$. By the righthand rule, $\mathbf{B}(1) = -\mathbf{k}$.

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Problem 4: (30 points)
Consider the lines:

$$\mathbf{r}_1(t) = \langle 5+2t, -t, 2+t\rangle$$

$$\mathbf{r}_2(t) = \langle 2+t, -1+2t, -2+3t\rangle$$

(a) Find the point of intersection of these two lines.
(b) Find the equation of the plane that contains both of these lines.
(c) Find the equation of a line that intersects both $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ but is orthogonal to both $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$.

SOLUTION:

(a) **(10 pts)** We can find the point of intersection by solving the system of equations $\mathbf{r}_1(t) = \mathbf{r}_2(s)$ for t and s.

$$x : 5+2t = 2+s$$
$$y : -t = -1+2s$$
$$z : 2+t = -2+3s$$

Thus, $t=-1$, $s=1$, and our point is given by $(3,1,1)$.

(b) **(10 pts)** To calculate the equation of a plane, we require a point on the plane and a vector normal to the plane.

Any point on either line will be sufficient. For simplicity, we choose $\mathbf{r}_1(0) = \langle 5, 0, 4 \rangle$.

For the normal vector, we cross $\mathbf{v}_1$ and $\mathbf{v}_2$.
$\mathbf{v}_1 = \langle 2, -1, 1 \rangle$
$\mathbf{v}_2 = \langle 1, 2, 3 \rangle$
Thus, $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = \langle -5, -5, 5 \rangle$

Our answer is therefore $\langle -5, -5, 5 \rangle \cdot \langle x-5, y, z-2 \rangle = 0$ (or an algebraically equivalent answer).

(c) **(10 pts)** To calculate the equation of a line, we require a point on the line and a vector parallel to the line.

The point is given by the point of intersection between the two lines. Our point from part (a) is given by $(3, 1, 1)$.

Calculating the vector parallel to our line as above, we have $\mathbf{r}_3(t) = \langle -5, -5, 5 \rangle t + \langle 3, 1, 1 \rangle$.

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Problem 5: (30 points)
A space probe passes through the point $(x, y, z) = (3, -2, 1)$ at time $t = \pi$. Its velocity at time t is given by

$$\mathbf{v}(t) = -3\cos^2 t \sin t\,\mathbf{j} + 3\sin^2 t \cos t\,\mathbf{k}$$

(a) Find the location of the space probe when $t = \pi/2$ (give your answer as a point).
(b) Find the curvature of the space probe's path when $t = \pi/4$.
(c) Let the point P_0 represent the point in space where the probe is located at time $t = 0$, and P_2 represent the point in space where the probe is located at time $t = \pi/2$. How much further does the probe travel to go from P_0 to P_2 than if it traveled between those points along a straight path?

SOLUTION:

(a) **(9 pts)** Integrating $\mathbf{v}(t)$ term by term we have
$\mathbf{r}(t) = C_1\mathbf{i} + \left(\cos^3 t + C_2\right)\mathbf{j} + \left(\sin^3 t + C_3\right)\mathbf{k}$.
We're given $\mathbf{r}(\pi) = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, which implies $C_1 = 3$, $C_2 = -1$ and $C_3 = 1$.
Thus $\mathbf{r}(t) = 3\mathbf{i} + \left(\cos^3 t - 1\right)\mathbf{j} + \left(\sin^3 t + 1\right)\mathbf{k}$.
$\implies \mathbf{r}(\pi/2) = 3\mathbf{i} - 1\mathbf{j} + 2\mathbf{k}$. Thus at time $t = \pi/2$ the space probe is at the point $(3, -1, 2)$.

(b) **(10 pts)**
Method 1: $\kappa = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$
$|\mathbf{r}'(t)| = |\mathbf{v}(t)| = \sqrt{0^2 + (-3\cos^2 t \sin t)^2 + (3\sin^2 t \cos t)^2} = \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t}$
$= \sqrt{9\cos^2 t \sin^2 t(\cos^2 t + \sin^2 t)} = 3|\cos t \sin t|$
$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \frac{0\mathbf{i} - 3\cos^2 t \sin t\,\mathbf{j} + 3\sin^2 t \cos t\,\mathbf{k}}{3|\cos t \sin t|} = 0\mathbf{i} - \cos t\,\mathbf{j} + \sin t\,\mathbf{k}$, for $0 < t < \pi/2$
$\implies \mathbf{T}'(t) = 0\mathbf{i} + \sin t\,\mathbf{j} + \cos t\,\mathbf{k} \implies |\mathbf{T}'(t)| = 1$
Thus $\kappa = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{1}{3|\cos t \sin t|}$
Thus when $t = \pi/4$, $\kappa = \frac{1}{3(\frac{1}{\sqrt{2}})(\frac{1}{\sqrt{2}})} = \frac{2}{3}$
Method 2:
$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$
$\mathbf{v}(\pi/4) = -\frac{3}{2\sqrt{2}}\mathbf{j} + \frac{3}{2\sqrt{2}}\mathbf{k}$ and $|\mathbf{v}(\pi/4)| = 3/2$
$\mathbf{a}(t) = \left(-3\cos^3 t + 6\cos t \sin^2 t\right)\mathbf{j} + \left(-3\sin^3 t + 6\sin t \cos^2 t\right)\mathbf{k}$
$\implies \mathbf{a}(\pi/4) = \frac{3}{2\sqrt{2}}\mathbf{j} + \frac{3}{2\sqrt{2}}\mathbf{k}$
$\mathbf{v}(\pi/4) \times \mathbf{a}(\pi/4) = -\frac{9}{4}\mathbf{i}$
$\implies |\mathbf{v}(\pi/4) \times \mathbf{a}(\pi/4)| = \frac{9}{4}$
$\implies \kappa = \frac{9/4}{(3/2)^3} = \frac{2}{3}$

(c) **(11 pts)**

Distance traveled by probe: $\int_0^{\pi/2} |\mathbf{v}(t)|dt = \int_0^{\pi/2} 3\cos t \sin t dt = \frac{3}{2}\sin^2 t|_0^{\pi/2} = \frac{3}{2}$
$\mathbf{r}(0) = 3\mathbf{i} + \mathbf{k} \implies$ at time $t = 0$, probe is at the point $(3, 0, 1)$.
The distance between the points $(3, 0, 1)$ and $(3, -1, 2)$ is $\sqrt{0^2 + (-1)^2 + 1^2} = \sqrt{2}$.
Thus the probe travels $3/2 - \sqrt{2}$ further.

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