Write your name below. This exam is worth 100 points. On each problem, you must show all your work to receive credit on that problem. You are allowed to use one page of notes. You cannot collaborate on the exam or seek outside help, nor can you use the recorded lectures, a calculator, any computational software, or material you find online.

Name:

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1. (26 points) Consider the two vectors:

$$\mathbf{v}_1 = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$$

$$\mathbf{v}_2 = \mathbf{i} + \mathbf{j} + \mathbf{k}$$

(a) (6 points) Are these vectors orthogonal? Are they parallel? Justify your answer.

Solution: We find that $\mathbf{v}_1 \cdot \mathbf{v}_2 = 4$, so the vectors are not orthogonal. We also see that they are not multiples of each other, $\mathbf{v}_1 \neq m\mathbf{v}_2$ for $m \in \mathbb{R}$, so they are not parallel either.

(b) (6 points) Find a vector that is orthogonal to both of these vectors.

Solution: By the properties of cross products, a vector orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$ is

$$\begin{aligned}\mathbf{v}_1 \times \mathbf{v}_2 &= (-1-2)\mathbf{i} + (2-3)\mathbf{j} + (3+1)\mathbf{k} \\ &= -3\mathbf{i} + -\mathbf{j} + 4\mathbf{k}.\end{aligned}$$

(c) (6 points) What is the area of the parallelogram formed by $\mathbf{v}_1$ and $\mathbf{v}_2$?.

Solution: The area of this parallelogram is

$$|\mathbf{v}_1 \times \mathbf{v}_2| = \sqrt{(-3)^2 + (-1)^2 + 4^2} = \sqrt{26}$$

(d) (8 points) If we add a third vector, $\mathbf{v}_3 = \mathbf{i} - 2\mathbf{j} - \mathbf{k}$, what is the volume of the parallepiped formed by these three vectors?

Solution: The volume is given by the absolute value of the triple scalar product:

$$|\mathbf{v}_3 \cdot (\mathbf{v}_1 \times \mathbf{v}_2)| = |-3 + 2 - 4| = 5$$

2. (26 points) Consider the plane given by

$$3x - 2y + z = 2$$

and the line given by the symmetric equations

$$\frac{x+1}{2} = y = 2 - z.$$

(a) (6 points) Find the point where the line intersects the plane.

Solution: We substitue the parameterized equations for this line:

$x = 2t - 1$
$y = t$
$z = 2 - t$

into the equation for the pane and solve for t:

$3(2t - 1) - 2(t) + (2 - t) = 2$
$6t - 3 - 2t + 2 - t = 2$
$t = 1$

Substituting this value of t back into our parameterized equations gives

$x = 2 - 1 = 1$
$y = 1$
$z = 2 - 1 = 1$

So the point is $(1, 1, 1)$.

(b) (6 points) Find the angle between the line and plane's normal vector.

Solution: We can read off the line's direction vector, $\mathbf{v} =< 2, 1, -1 >$, and the normal vector of the plane, $\mathbf{n} =< 3, -2, 1 >$, directly from their equations. We then find that

$\mathbf{v} \cdot \mathbf{n} = 3$
$|\mathbf{v}| = \sqrt{6}$
$|\mathbf{n}| = \sqrt{14}$

$$\theta = \cos^{-1}\left(\frac{\mathbf{v} \cdot \mathbf{n}}{|\mathbf{v}||\mathbf{n}|}\right) = \cos^{-1}\left(\frac{3}{\sqrt{84}}\right) = \cos^{-1}\left(\frac{1}{2}\sqrt{\frac{3}{7}}\right)$$

(c) (6 points) Show that the point A$(3, 2, 0)$ is on the line.

Solution: We substitute these values into the symmetric equations and find they are valid:

$$\frac{3+1}{2} = 2 = 2 - 0,$$

so $(3, 2, 0)$ is on the line.

(d) (8 points) Determine how far the point A is from the plane.

Solution: The point $(1, 1, 1)$ is on the plane, so we find the vector from this point to $(3, 2, 0)$. The absolute value of the scalar component of this vector along the plane's normal vector is the distance :

$\mathbf{u} =< 2, 1, -1 >$

$$d = \frac{|\mathbf{n} \cdot \mathbf{u}|}{|\mathbf{n}|} = \frac{3}{\sqrt{14}}$$

3. (20 points) A quadric surface is defined by the equation

$$2x^2 - 4x + y^2 - z^2 - 2z = 0$$

(a) (8 points) Classify the quadric surface and state its orientation.

Solution: We must complete the squares for x and z to get the quadric in a standard form:

$2(x^2 - 2x) + y^2 - (z^2 + 2z) = 0$
$2(x^2 - 2x + 1) + y^2 - (z^2 + 2z + 1) = 2 - 1$
$2(x - 1)^2 + y^2 - (z + 1)^2 = 1$

We see this is a hyperboloid of one sheet oriented parallel to the z-axis.

(b) (6 points) Find the center of the surface.

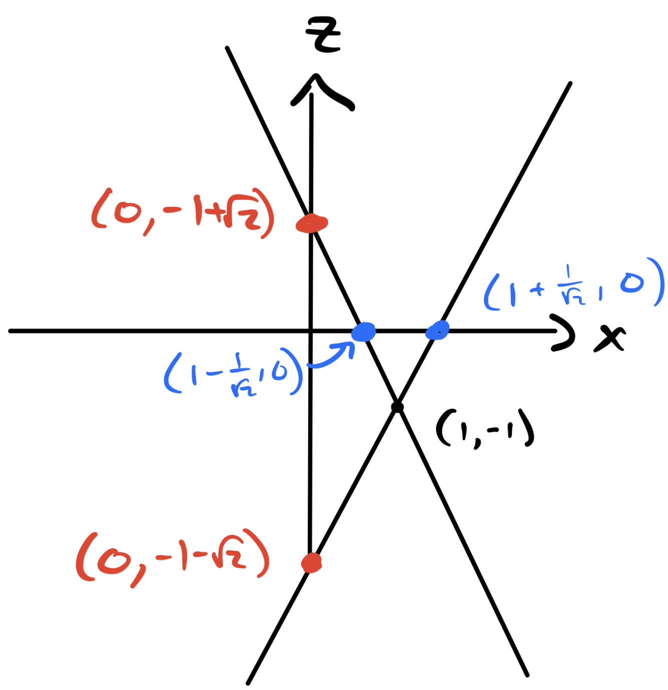
Solution: Reading offsets from the standard form above, we see that the center is at $(1, 0, -1)$

(c) (6 points) Sketch the trace for $y = 1$. Label all x and z intercepts.

Solution: When $y = 1$ we have

$2(x - 1)^2 = (z + 1)^2$
$z + 1 = \pm\sqrt{2}(x - 1)$
$z - (-1) = \pm\sqrt{2}(x - 1)$

So we have two lines which intersect at $(1, -1)$ in the xz plane with slopes $\pm\sqrt{2}$:



4. (28 points) An ion moving in a magnetic field starts at the origin and follows a path given by the position vector

$$\mathbf{r}(t) = \left\langle \frac{t}{2} + \sin(t), \cos(t) - 1, \frac{3t}{8} \right\rangle$$

(a) (8 points) After the ion has moved 3π in the z direction, the ion collides with a neutral atom. Where does this collision occur? How far from the origin does the collision occur?

Solution: We know that the ion's position vector has a z component of 3π, so we use this to find the time t of the collision and substitute it into the other components to find the position:

$$\frac{3t}{8} = 3\pi$$

$$t = 8\pi$$

$$\mathbf{r}(8\pi) = \left\langle \frac{8\pi}{2} + \sin(8\pi), \cos(8\pi) - 1, \frac{(3)(8\pi)}{8} \right\rangle = \langle 4\pi, 0, 3\pi \rangle$$

The distance from the origin is

$$d = \sqrt{(4\pi)^2 + (3\pi)^2} = \sqrt{25\pi^2} = 5\pi$$

(b) (8 points) Find the velocity vector and acceleration vector of the ion. What is their dot product?

Solution: We simply differentiate to find these vectors:

$$\mathbf{v}(t) = \mathbf{r}'(t) = \left\langle \frac{1}{2} + \cos(t), -\sin(t), \frac{3}{8} \right\rangle$$

$$\mathbf{a}(t) = \mathbf{v}'(t) = \langle -\sin(t), -\cos(t), 0 \rangle$$

And there dot product is

$$\mathbf{v}(t) \cdot \mathbf{a}(t) = (\frac{1}{2} + \cos(t))(-\sin(t)) + (-\sin(t))(-\cos(t)) = -\frac{1}{2}\sin(t)$$

(c) (6 points) Set up, but do not evaluate, the integral to determine how far the ion has moved along it's path before the collision.

$$\begin{aligned} L = \int_0^{8\pi} |\mathbf{v}(t)| dt &= \int_0^{8\pi} \sqrt{(\frac{1}{2} + \cos(t))^2 + (-\sin(t))^2 + (\frac{3}{8})^2} dt \\ &= \int_0^{8\pi} \sqrt{\frac{1}{4} + \cos(t) + \cos^2(t) + \sin^2(t) + \frac{9}{64}} dt \\ &= \int_0^{8\pi} \sqrt{\frac{1}{4} + \cos(t) + 1 + \frac{9}{64}} dt \\ &= \int_0^{8\pi} \sqrt{\cos(t) + \frac{89}{64}} dt \end{aligned}$$

(d) (6 points) In the collision, the ion becomes a neutral atom and continues traveling in a straight line afterwards. Find an equation for this line.

Solution: The line contains the point $\mathbf{r}(8\pi)$ and has a direction vector $\mathbf{r}'(8\pi)$:

$$\mathbf{r}'(8\pi) = \mathbf{v}(8\pi) = \left\langle \frac{3}{2}, 0, \frac{3}{8} \right\rangle$$

so the line is given by

$$\mathbf{u}(s) = \langle 4\pi, 0, 3\pi \rangle + \left\langle \frac{3}{2}, 0, \frac{3}{8} \right\rangle s$$