1. (28 pts) The height of a hill in meters is given by

$$h(x,y) = 9 - \sqrt{3 + x^2 + y^2}, \quad x^2 + y^2 \le 78.$$

Gamma Goat is walking on the hill and has reached the point $P\,(2,3,5)$.

(a) Gamma Goat spots Beta Bee which is hovering at point $Q(4,4,7)$. Find a vector equation for the line segment connecting P and Q.

(b) If Gamma Goat decides to descend the hill as quickly as possible, in which direction should the goat walk? Express your answer in the form of a vector $p\mathbf{i} + q\mathbf{j}$.

(c) Gamma Goat chooses instead to walk across the hill, maintaining its elevation above the ground. Find a vector equation for this path in the form $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$.

(d) The shape of the hill matches which quadric surface?

Solution:

(a) $\mathbf{PQ} = \langle 2,1,2\rangle$, so the line segment connecting the two points is

$$\mathbf{s}(t) = \langle 2,3,5\rangle + t\langle 2,1,2\rangle = \langle 2+2t, 3+t, 5+2t\rangle, \quad 0 \le t \le 1.$$

(b) The direction of steepest descent is $-\nabla h(2,3)$. Given

$$\nabla h(x,y) = \left\langle \frac{-x}{\sqrt{3+x^2+y^2}}, \frac{-y}{\sqrt{3+x^2+y^2}} \right\rangle, \text{ then}$$

$$-\nabla h(2,3) = -\left\langle \frac{-2}{\sqrt{3+2^2+3^2}}, \frac{-3}{\sqrt{3+2^2+3^2}} \right\rangle = \left\langle \frac{1}{2}, \frac{3}{4} \right\rangle.$$

(c) The level curve at P has equation

$$h(x,y) = 5 = 9 - \sqrt{3+x^2+y^2} \implies \sqrt{3+x^2+y^2} = 4 \implies x^2+y^2 = 13.$$

Applying the parametrization $x(t) = \sqrt{13}\cos t$, $y(t) = \sqrt{13}\sin t$, a vector equation for the curve is

$$\mathbf{r}(t) = \left(\sqrt{13}\cos t\right)\mathbf{i} + \left(\sqrt{13}\sin t\right)\mathbf{j} + 5\mathbf{k}.$$

For the parametric interval, because point P has coordinates $x = 2$ and $y = 3$, then at P, the value of t is $\cos^{-1}\left(\frac{2}{\sqrt{13}}\right) = \sin^{-1}\left(\frac{3}{\sqrt{13}}\right)$. The goat can walk in either direction, so let the parametric interval be $t \ge \cos^{-1}\left(\frac{2}{\sqrt{13}}\right)$ or $t \le \cos^{-1}\left(\frac{2}{\sqrt{13}}\right)$.

(d) The equation corresponds to $(z-9)^2 - x^2 - y^2 = 3$ which is a hyperboloid of 2 sheets.

2. (15 pts) Zeta is building an open-top wooden rectangular box with a square base. The volume of the box will be 4000 cm^3. **Use Lagrange multipliers** to find the box dimensions that will minimize the amount of wood needed.

Solution:

Let the dimensions of the square base be x by x cm for $x > 0$, and the height of the box be y cm for $y > 0$. Then the volume of the box is $V(x, y) = x^2y$, and the amount of wood needed equals the surface area $S(x, y) = x^2 + 4xy$. We wish to minimize $S(x, y)$ given the constraint $V(x, y) = 4000$.

$$S(x,y) = x^2 + 4xy \qquad \nabla S = \langle 2x + 4y, 4x \rangle$$
$$V(x,y) = x^2y \qquad \nabla V = \langle 2xy, x^2 \rangle$$

$\nabla S = \lambda \nabla V$ implies

$$\lambda(2xy) = 2x + 4y \qquad \Longrightarrow \lambda = \frac{x + 2y}{xy}$$
$$\lambda x^2 = 4x \qquad \Longrightarrow \lambda = \frac{4}{x}.$$

Setting the two λ expressions equal yields

$$\begin{aligned} \frac{x + 2y}{xy} &= \frac{4}{x} \\ \frac{x + 2y}{y} &= \frac{4}{1} \\ x + 2y &= 4y \\ x &= 2y \end{aligned}$$

Substituting $x = 2y$ into the $V(x, y) = x^2y = 4000$ constraint gives

$$(2y)^2y = 4000 \implies y^3 = 1000 \implies y = 10$$

and $x = 2y = 20$. Therefore the largest box will have a base measuring 20 by 20 cm and a height of 10 cm.

3. (16 pts) The joint probability density function for random variables X and Y is

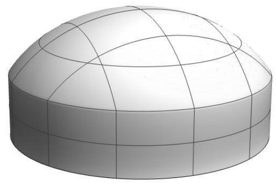
$$f(x, y) = \begin{cases} 3e^{-x}e^{-3y} & x > 0,\ y > 0 \\ 0 & \text{otherwise.} \end{cases}$$

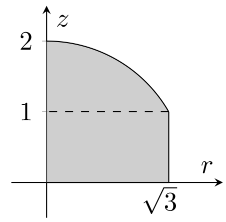
Compute $P(X < Y) = \int_0^\infty \int_0^y 3e^{-x}e^{-3y}\, dx\, dy$.

Solution:

$$\begin{aligned}
\int_0^\infty \int_0^y 3e^{-x}e^{-3y}\, dx\, dy &= 3\int_0^\infty e^{-3y} \int_0^y e^{-x}\, dx\, dy \\
&= 3\int_0^\infty e^{-3y} \left[-e^{-x}\right]_0^y\, dy \\
&= 3\int_0^\infty e^{-3y} \left(-e^{-y} + 1\right) dy \\
&= 3\int_0^\infty \left(-e^{-4y} + e^{-3y}\right) dy \\
&= 3\lim_{t\to\infty} \left[\frac{1}{4}e^{-4y} - \frac{1}{3}e^{-3y}\right]_0^t \\
&= 3\lim_{t\to\infty} \left(\frac{1}{4}e^{-4t} - \frac{1}{3}e^{-3t} - \left(\frac{1}{4} - \frac{1}{3}\right)\right) \\
&= 3\left(0 - 0 - \frac{1}{4} + \frac{1}{3}\right) = \frac{1}{4}
\end{aligned}$$

4. (27 pts) A cylindrical silo with a spherical top has the cross-section shown at right for an arbitrary value of θ.





Set up (but do not evaluate) integral(s) to calculate the volume of the silo

(a) using rectangular coordinates in the order $dz\,dy\,dx$
(b) using cylindrical coordinates in the order $dz\,dr\,d\theta$
(c) using spherical coordinates in the order $d\rho\,d\phi\,d\theta$.

Solution:

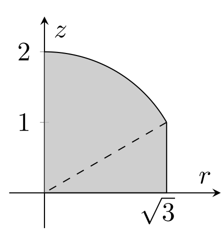
(a) The spherical top has equation $x^2 + y^2 + z^2 = 4$ for $r^2 = x^2 + y^2 \le 3$. Thus z extends from 0 to $\sqrt{4 - x^2 - y^2}$. The projection of the silo onto the xy-plane is a circle of radius $\sqrt{3}$ centered at the origin. Therefore the volume in rectangular coordinates is

$$\int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}} \int_0^{\sqrt{4-x^2-y^2}} dz\,dy\,dx.$$

(b) For the spherical top, substitute $r^2 = x^2 + y^2$ to obtain the equation $r^2 + z^2 = 4$. Thus z extends from 0 to $\sqrt{4 - r^2}$. In the xy=plane, θ extends from 0 to 2π, and r extends from 0 to $\sqrt{3}$. Therefore the volume in cylindrical coordinates is

$$\int_0^{2\pi} \int_0^{\sqrt{3}} \int_0^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta.$$

(c) Split the cross-section into two regions, dividing diagonally as shown.



In the upper region, the spherical top has equation $\rho = 2$ and ϕ extends from 0 to $\tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}$. In the lower region, ϕ extends from $\frac{\pi}{3}$ to $\frac{\pi}{2}$ and ρ extends from 0 to $r = \sqrt{3} \implies \rho \sin\phi = \sqrt{3} \implies \rho = \sqrt{3}\csc\phi$. Therefore the volume in spherical coordinates is

$$\int_0^{2\pi} \int_0^{\pi/3} \int_0^{2} \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta + \int_0^{2\pi} \int_{\pi/3}^{\pi/2} \int_0^{\sqrt{3}\csc\phi} \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta.$$

5. (16 pts) Use Gaussian elimination and back substitution to solve the linear system.

$$\begin{aligned} -x \quad\quad + 2z &= 3 \\ 3x + y - z &= 2 \\ x + y + z &= -2 \end{aligned}$$

Solution: Row reduce the augmented matrix $[\,\mathbf{A} \mid \mathbf{b}\,]$ to REF.

$$\left[\begin{array}{rrr|r} -1 & 0 & 2 & 3 \\ 3 & 1 & -1 & 2 \\ 1 & 1 & 1 & -2 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -2 & -3 \\ 0 & 1 & 5 & 11 \\ 0 & 1 & 3 & 1 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -2 & -3 \\ 0 & 1 & 5 & 11 \\ 0 & 0 & -2 & -10 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -2 & -3 \\ 0 & 1 & 5 & 11 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

The reduced system corresponds to these equations:

$$\begin{aligned} x \quad - 2z &= -3 \\ y + 5z &= 11 \\ z &= 5 \end{aligned}$$

Using back substitution, we find that $z = 5$, $y = -14$, and $x = 7$.

6. (17 pts) Solve the linear system by finding the inverse of the coefficient matrix.

$$\begin{aligned} \tfrac{x}{2} \quad\; &+ \; z = \;\;1 \\ 2x - y \quad\; &\quad\; = -3 \\ x \quad\; &+ 3z = \;\;1 \end{aligned}$$

Solution:

$$\mathbf{A} = \begin{bmatrix} \frac{1}{2} & 0 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 3 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$$

Find $\mathbf{A}^{-1}$ by row reducing the augmented matrix $[\,\mathbf{A} \mid \mathbf{I}\,]$ to produce the form $[\,\mathbf{I} \mid \mathbf{A}^{-1}\,]$.

$$\left[\begin{array}{ccc|ccc} \frac{1}{2} & 0 & 1 & 1 & 0 & 0 \\ 2 & -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 3 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 2 & 0 & 0 \\ 0 & -1 & -4 & -4 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 2 & 0 & 0 \\ 0 & 1 & 4 & 4 & -1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 6 & 0 & -2 \\ 0 & 1 & 0 & 12 & -1 & -4 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]$$

Then multiply $\mathbf{A}^{-1}$ by $\mathbf{b}$.

$$\mathbf{A}^{-1}\mathbf{b} = \begin{bmatrix} 6 & 0 & -2 \\ 12 & -1 & -4 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 11 \\ -1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

The solution is $x = 4$, $y = 11$, and $z = -1$.

7. (15 pts) Consider this linear system in variables x and y. Find nonzero constants a, b, c, and d to produce each of the following results. (There are multiple possible answers.)

$$\begin{aligned} -3x + 2y &= a \\ 3x - 2y &= b \\ 6x - cy &= d \end{aligned}$$

(a) The system has no solutions.

(b) The system has a unique solution.

(c) The system has infinitely many solutions.

Solution:

(a) There will be no solutions if the first two equations represent parallel lines, when $b \neq -a$. For example:

$$\begin{aligned} -3x + 2y &= 1 \\ 3x - 2y &= 4 \\ 6x - cy &= d \end{aligned}$$

(b) There will be a unique solution if the first two equations represent the same line and the third equation represents a different line, when $b = -a$ and $c \neq 4$. For example:

$$\begin{aligned} -3x + 2y &= 1 \\ 3x - 2y &= -1 \\ 6x - y &= d \end{aligned}$$

(c) There will be infinitely many solutions if the three equations represent the same line, when $b = -a$. $c = 4$, and $d = 2b$. For example:

$$\begin{aligned} -3x + 2y &= 1 \\ 3x - 2y &= -1 \\ 6x - 4y &= -2 \end{aligned}$$

8. (16 pts) Given the points $(0,2)$, $(-1,0)$, $(-2,1)$, solve a linear system to find the least-squares line of best fit.

Solution: Let the line be $y = c_0 + c_1 x$. We wish to find constants c_0 and c_1 such that the least-squares error is minimized for

$$\begin{aligned} c_0 \quad\quad &\approx 2 \\ c_0 - \quad c_1 &\approx 0 \\ c_0 - 2c_1 &\approx 1. \end{aligned}$$

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 1 & -2 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

We wish to solve $\mathbf{A}^T\mathbf{A}\mathbf{v} = \mathbf{A}^T\mathbf{b}$. Given $\mathbf{A}^T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \end{bmatrix}$,

$$\mathbf{A}^T\mathbf{A} = \begin{bmatrix} 3 & -3 \\ -3 & 5 \end{bmatrix} \quad \text{and} \quad \mathbf{A}^T\mathbf{b} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}.$$

Row reducing the augmented matrix to RREF yields

$$\left[\begin{array}{cc|c} 3 & -3 & 3 \\ -3 & 5 & -2 \end{array}\right] \quad \to \quad \left[\begin{array}{cc|c} 1 & 0 & \frac{3}{2} \\ 0 & 1 & \frac{1}{2} \end{array}\right],$$

so $c_0 = \frac{3}{2}$, $c_1 = \frac{1}{2}$, and the least-squares line is $y = \frac{3}{2} + \frac{1}{2}x$.