1. (16 pts) The area of a garden is given by

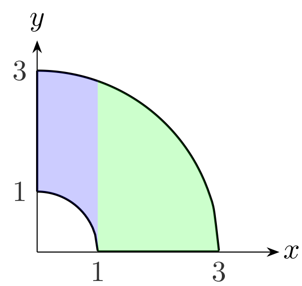
$$\int_0^1 \int_{\sqrt{1-x^2}}^{\sqrt{9-x^2}} dy\,dx + \int_1^3 \int_0^{\sqrt{9-x^2}} dy\,dx.$$

   (a) Sketch the shape of the garden.

   (b) Use polar coordinates to combine the two integrals into one double integral. Do not evaluate the result.

Solution:

(a) The region lies between circles of radius 1 and 3 in Q1.



(b) An equivalent integral in polar coordinates is $\displaystyle\int_0^{\pi/2}\int_1^3 r\,dr\,d\theta$.

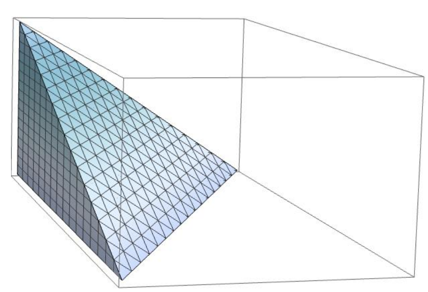
2. (14 pts) A tetrahedron in the first octant is bounded by the coordinate planes and the surface $x + 2y + 3z = 6$.

   (a) Sketch and shade the projection of the region onto the yz-plane. Label the intercepts.

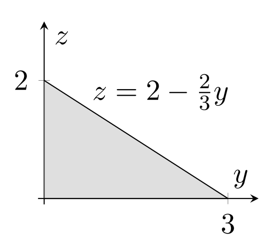
   (b) Set up (but do not evaluate) a triple integral to find the volume of the region using rectangular coordinates in the order $dx\,dz\,dy$.

Solution:

The solid is a tetrahedron with vertices at $(0,0,0)$, $(6,0,0)$, $(0,3,0)$, and $(0,0,2)$.



(a) The projection of the solid onto the yz-plane corresponds to the region bounded by $2y + 3z = 6$ and the positive y and z axes.



(b) In the x direction, the bounds extend from 0 to the plane $x = 6 - 2y - 3z$. Using the sketched region in the yz-plane, the integral is

$$\int_0^3 \int_0^{2-2y/3} \int_0^{6-2y-3z} dx\, dz\, dy.$$

3. (23 pts) Consider the solid with volume $V = \int_{-2}^{2} \int_0^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} dz\, dy\, dx.$

(a) Sketch a cross-section of the solid in the rz–plane (that is, a half-plane of constant θ). Label the intercepts.

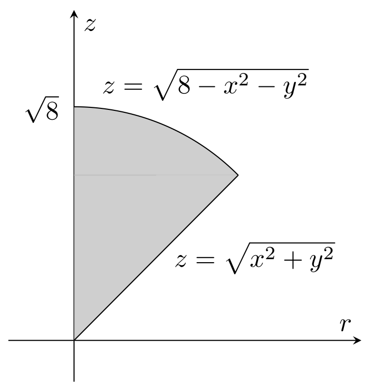
(b) Set up (but do not evaluate) an equivalent integral using

i. cylindrical coordinates in the order $dz\, dr\, d\theta$.

ii. spherical coordinates in the order $d\rho\, d\phi\, d\theta$.

Solution:

(a) The solid has the shape of an ice cream cone above the upper xy-plane. Below is a cross-section of the solid in the rz-plane.



(b) i. The intersection of the cone and the sphere is

$$\sqrt{x^2 + y^2} = \sqrt{8 - x^2 - y^2} \implies x^2 + y^2 = 4, \quad y \geq 0.$$

The projection of the 3D region onto the xy-plane is a semicircular region of radius 2 above the x-axis. In polar coordinates, the region in the xy-plane is defined by $0 \leq \theta \leq \pi$ and $0 \leq r \leq 2$. In the z direction, the solid extends from the cone $z = \sqrt{x^2 + y^2} = \sqrt{r^2} = r$ to the sphere $z = \sqrt{8 - x^2 - y^2} = \sqrt{8 - r^2}$. Therefore the integral in cylindrical coordinates is

$$\int_0^{\pi} \int_0^2 \int_r^{\sqrt{8-r^2}} r\, dz\, dr\, d\theta.$$

ii. In spherical coordinates, the equation for the sphere is $\rho = \sqrt{8} = 2\sqrt{2}$. The equation for the cone is

$$z = r \implies \rho \cos\phi = \rho \sin\phi \implies \tan\phi = 1 \implies \phi = \tfrac{\pi}{4}.$$

The angle θ ranges from 0 to π. Therefore the integral in spherical coordinates is

$$\int_0^{\pi} \int_0^{\pi/4} \int_0^{2\sqrt{2}} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.$$

4. (15 pts) Consider the integral

$$\int_0^1 \int_{-3x}^{x} (x-y)(3x+y)^2 \, dy \, dx.$$

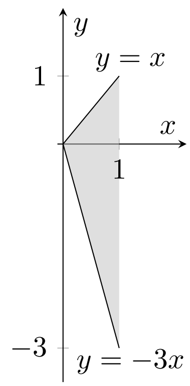
(a) Sketch the region of integration in the xy-plane.

(b) Let $u = x - y$ and $v = 3x + y$. Sketch the transformed region in the uv-plane.

(c) Set up an equivalent integral in the uv-plane. Do not evaluate the integral.

Solution:

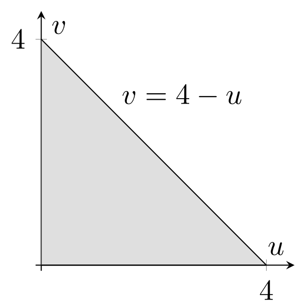
(a) The xy region is a triangle bounded by $y = x$, $y = -3x$, and $x = 1$.



(b) If $u = x - y$ and $v = 3x + y$, then $x = \frac{1}{4}(u+v)$ and $y = \frac{1}{4}(v - 3u)$.

xy boundaries	uv boundaries	uv equations
$x = 1$	$\frac{u+v}{4} = 1$	$v = 4 - u$
$y = x$	$\frac{1}{4}(v-3u) = \frac{1}{4}(u+v)$	$u = 0$
$y = -3x$	$\frac{1}{4}(v-3u) = -\frac{3}{4}(u+v)$	$v = 0$

The corresponding uv region is a triangle bounded by $u = 0$, $v = 0$, and $v = 4 - u$.



(c) The Jacobian is

$$J(u,v) = \begin{vmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{3}{4} & \frac{1}{4} \end{vmatrix} = \frac{1}{4}.$$

An equivalent integral in the uv-plane is

$$\int_0^4 \int_0^{4-u} \frac{1}{4} u v^2 \, dv \, du \quad \text{or} \quad \int_0^4 \int_0^{4-v} \frac{1}{4} u v^2 \, du \, dv.$$

5. (18 pts) Random variable X has the probability density function

$$f(x) = \begin{cases} \dfrac{a}{x^n} & \text{if } x \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

for some constant a and $n > 1$.

(a) Find a and express your answer in terms of n.

(b) Set up (but do not evaluate) an integral to calculate $(X > k)$ for $k \geq 1$.

Solution:

(a)

$$\begin{aligned} \int_1^\infty ax^{-n}\,dx &= a \cdot \lim_{t\to\infty} \left[\frac{x^{-n+1}}{-n+1} \right]_1^t \\ &= \frac{a}{-n+1} \cdot \lim_{t\to\infty} \left(t^{-n+1} - 1\right) \\ &= \frac{a}{-n+1}(0-1) = \frac{a}{n-1} \end{aligned}$$

Because $\displaystyle\int_1^\infty ax^{-n}\,dx = \frac{a}{n-1} = 1$, the value of a is $n-1$.

(b) $\displaystyle\int_k^\infty (n-1)x^{-n}\,dx$

6. (14 pts) Let $B = \begin{bmatrix} -1 & 3 & 0 \\ 2 & 0 & 1 \end{bmatrix}$.

(a) Find $B^T B$.

(b) Let A be any matrix and let $C = A^T A$. Prove that C is symmetric.

Solution:

(a) $B^T B = \begin{bmatrix} -1 & 2 \\ 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & 0 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -3 & 2 \\ -3 & 9 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

(b) Matrix C is symmetric if $C^T = C$. Using the properties $(BA)^T = A^T B^T$ and $\left(A^T\right)^T = A$, we find that

$$C^T = \left(A^T A\right)^T = A^T \left(A^T\right)^T = A^T A = C.$$

Therefore C is symmetric.

Alternate Solution:

If A is an $m \times n$ matrix, then A^T is $n \times m$ and $C = A^T A$ is an $n \times n$ square matrix.
In matrix $C = A^T A$, for all $1 \leq i, j \leq n$,

$$\begin{aligned} c_{ij} &= \left(i\text{th row of } A^T\right) \cdot (j\text{th col of } A) = (i\text{th col of } A) \cdot (j\text{th col of } A) \\ c_{ji} &= \left(j\text{th row of } A^T\right) \cdot (i\text{th col of } A) = (j\text{th col of } A)) \cdot (i\text{th col of } A)\,. \end{aligned}$$

By the commutative property of dot product, $c_{ij} = c_{ji}$, so C is a symmetric matrix.