1. (18 pts) The following two problems are not related.

   (a) Let
   $$w = \frac{2}{x} + \ln(yz), \quad x = r\sin r, \quad y = \frac{r}{2s}, \quad z = \frac{s^2}{r}.$$
   Find $\partial w/\partial s$. Express your answer in terms of r and s.

   (b) Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x^2y}{3x^4+4y^2}$ does not exist.

Solution:

(a)

$$\begin{aligned}
\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial s} \\
&= \frac{1}{y}\left(\frac{-r}{2s^2}\right) + \frac{1}{z}\left(\frac{2s}{r}\right) \\
&= \frac{-r}{2s^2y} + \frac{2s}{rz} \\
&= \frac{-r}{2s^2}\cdot\frac{2s}{r} + \frac{2s}{r}\cdot\frac{r}{s^2} \\
&= -\frac{1}{s} + \frac{2}{s} = \frac{1}{s}
\end{aligned}$$

(b) Approaching $(0,0)$ along the x-axis,

$$\lim_{(x,0)\to(0,0)} \frac{2x^2y}{3x^4+4y^2} = \lim_{(x,0)\to(0,0)} \frac{0}{3x^4} = 0.$$

Similarly the limit equals 0 approaching the origin along the y-axis.

Approaching $(0,0)$ along the line $y = x$, the result is the same:

$$\lim_{(x,x)\to(0,0)} \frac{2x^2y}{3x^4+4y^2} = \lim_{(x,x)\to(0,0)} \frac{2x^3}{3x^4+4x^2} = \lim_{(x,x)\to(0,0)} \frac{2x}{3x^2+4} = \frac{0}{4} = 0.$$

Approaching $(0,0)$ along the curve $y = x^2$, however,

$$\lim_{(x,x^2)\to(0,0)} \frac{2x^2y}{3x^4+4y^2} = \lim_{(x,x^2)\to(0,0)} \frac{2x^4}{3x^4+4x^4} = \frac{2}{7} \neq 0.$$

Therefore the limit does not exist.

2. (24 pts) Let $f(x, y) = x^2 y$.

(a) Find the rate of change of f at $Q(-1, 3)$ in the direction toward the origin.

(b) Find a unit vector tangent to the level curve $f(x, y) = 3$ at Q.

(c) What is the greatest possible rate of change of f at Q?

(d) Find a vector normal to the surface $z = f(x, y)$ at Q.

Solution:

(a)

$$\nabla f(x, y) = \langle 2xy, x^2 \rangle \implies \nabla f(-1, 3) = \langle -6, 1 \rangle$$

Let O represent the origin and let the direction vector $\mathbf{u} = \dfrac{\mathbf{QO}}{|\mathbf{QO}|} = \dfrac{\langle 1, -3 \rangle}{\sqrt{10}}$. Then

$$D_{\mathbf{u}} f(-1, 3) = \nabla f(-1, 3) \cdot \mathbf{u} = \langle -6, 1 \rangle \cdot \frac{\langle 1, -3 \rangle}{\sqrt{10}} = \frac{-9}{\sqrt{10}}.$$

(b) The gradient vector $\nabla f(-1, 3) = \langle -6, 1 \rangle$ is orthogonal to the level curve at Q, so a tangent vector is $\langle 1, 6 \rangle$ or $\langle -1, -6 \rangle$. The corresponding unit vector is $\dfrac{\langle 1, 6 \rangle}{\sqrt{37}}$ or $\dfrac{\langle -1, -6 \rangle}{\sqrt{37}}$.

(c) The greatest possible rate of change is $|\nabla f(-1, 3)| = |\langle -6, 1 \rangle| = \sqrt{37}$.

(d) Let $F(x, y, z) = f(x, y) - z$. Then $\nabla F(x, y, z) = \langle f_x, f_y, -1 \rangle$ and a normal vector to the surface at Q will be $\nabla F(-1, 3, 3) = \langle -6, 1, -1 \rangle$ or $\langle 6, -1, 1 \rangle$.

3. (26 pts) Let $g(x,y) = x\sin(2y)$.

(a) Find all critical points of g in the open region $R = \left\{(x,y) \mid |x| < \frac{\pi}{2},\ |y| < \frac{\pi}{2}\right\}$. Use the Second Derivatives Test to classify the points.

(b) Use Taylor series to find a quadratic approximation of g at $(0,0)$.

(c) Find the maximum error in the quadratic approximation of g for $|x| \leq 0.1,\ |y| \leq 0.1$. You may leave the final answer unsimplified.

Solution:

(a)

$$g_x = \sin(2y) \quad g_y = 2x\cos(2y)$$

The only critical point in the given region where $g_x = 0$ and $g_y = 0$ is the origin $(0,0)$.

$$\begin{aligned} g_{xx} &= 0 & g_{xx}(0,0) &= 0 \\ g_{yy} &= -4x\sin(2y) & g_{yy}(0,0) &= 0 \\ g_{xy} &= 2\cos(2y) & g_{xy}(0,0) &= 2 \end{aligned}$$

Applying the Second Derivatives Test, $D = 0 - g_{xy}(0,0)^2 = -4 < 0$. Therefore there is a saddle point at $(0,0)$.

(b) At $(0,0)$, the values of $g(0,0)$, $g_x(0,0)$, and $g_y(0,0)$ are all zero, so the quadratic approximation of g is

$$\begin{aligned} Q(x,y) &= g(0,0) + g_x(0,0)x + g_y(0,0)y \\ &\quad + \tfrac{1}{2}\left[g_{xx}(0,0)x^2 + 2g_{xy}(0,0)xy + g_{yy}(0,0)y^2\right] \\ &= 0 + 0 + 0 + \tfrac{1}{2}\left(0 + 4xy + 0\right) = 2xy. \end{aligned}$$

(c)

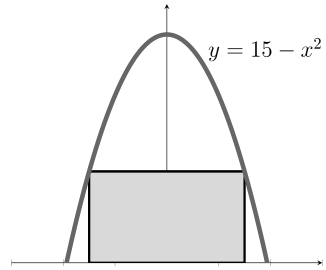
$$g_{xxx} = 0 \quad g_{xxy} = 0 \quad g_{xyy} = -4\sin(2y) \quad g_{yyy} = -8x\cos(2y)$$

Because $|\sin(2y)| \leq 1$ and $|\cos(2y)| \leq 1$, we have $|g_{xyy}| \leq 4 \cdot 1 = 4$ and $|g_{yyy}| \leq 8(0.1)(1) = 0.8$, so an upper bound for the third order partial derivatives is $M = 4$. Therefore an error bound for the quadratic approximation for $|x| \leq 0.1,\ |y| \leq 0.1$ is

$$\begin{aligned} |E(x,y)| &\leq \tfrac{1}{3!}M\left(|x| + |y|\right)^3 \\ &\leq \tfrac{4}{6}\left(0.1 + 0.1\right)^3 \\ &= \tfrac{2}{3}(0.2)^3 = \tfrac{0.016}{3} \approx 0.005. \end{aligned}$$

Note: Other choices for M are acceptable. Because $\sin\theta \approx \theta$, we have $|g_{xyy}| \leq 4\sin(0.2) \approx 4(0.2) = 0.8$. Another reasonable upper bound is $M = 1$.

4. (20 pts) An archway has the shape of the parabola $y = 15 - x^2$ for $y \geq 0$. Use Lagrange multipliers to determine the width and height in units of the largest rectangular box that will fit through the archway.



Solution:

The largest rectangular cross-section of the box will meet the parabola at its upper corners. Let (x, y) be the point on the curve corresponding to the upper right corner of the rectangle with $x, y > 0$. Then the rectangle will have a width of $2x$ and a height of y. We wish to maximize the rectangular area $f(x, y) = 2xy$. The point (x, y) must satisfy the equation $y = 15 - x^2$, so let the constraint be $g(x, y) = x^2 + y = 15$.

$$f(x, y) = 2xy \qquad\qquad \nabla f = \langle 2y, 2x \rangle$$
$$g(x, y) = x^2 + y \qquad\qquad \nabla g = \langle 2x, 1 \rangle$$

$\nabla f = \lambda \nabla g$ implies

$$\lambda(2x) = 2y \implies \lambda = \frac{y}{x}$$
$$\lambda = 2x.$$

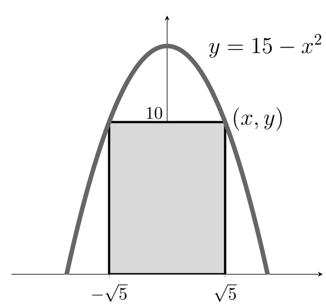
Setting the λ expressions equal yields

$$\frac{y}{x} = 2x \implies y = 2x^2.$$

Substituting $y = 2x^2$ into the $g(x, y) = 15$ constraint gives

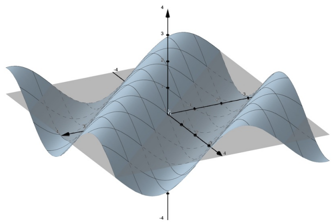
$$x^2 + 2x^2 = 15 \implies 3x^2 = 15 \implies x = \sqrt{5}$$

because $x > 0$. Then $y = 2x^2 = 10$. Therefore the largest box will have a width of $2\sqrt{5}$ units and a height of 10 units.

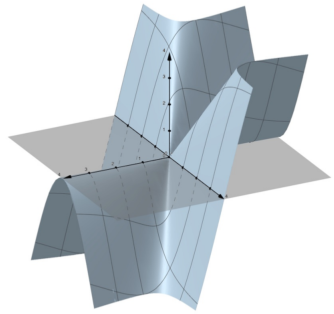


5. (12 pts) Match the three surfaces to their contour plots. No explanation is necessary. (For each surface, the first octant is facing toward the front.)

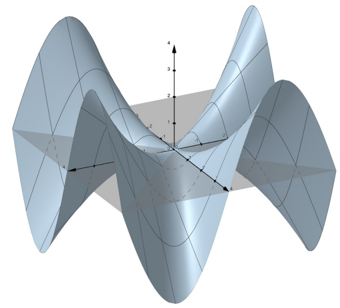
Surface 1



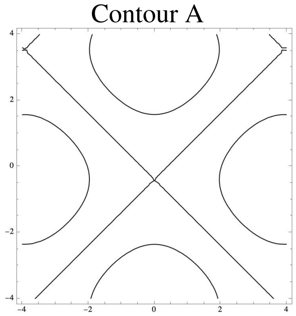
Surface 2



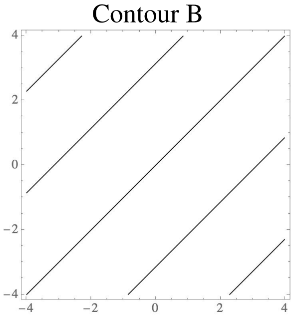
Surface 3



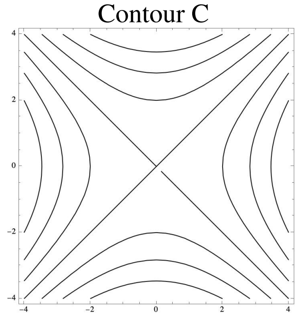
Contour A



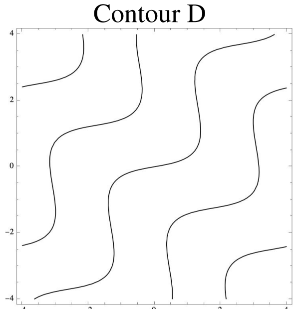
Contour B



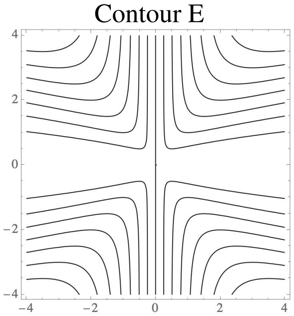
Contour C



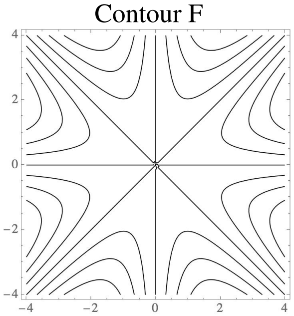
Contour D



Contour E



Contour F



Solution:

Surface 1: Contour B
Surface 2: Contour E
Surface 3: Contour F