1. (24 pts) Consider the points $P(0, 9, 2)$ and $Q(-2, 10, 5)$.

   (a) Find an equation for the set of points equidistant from point P and the plane $z = 7$. You may leave your answer unsimplified.

   (b) Let $\mathbf{v}$ equal the vector $\overrightarrow{PQ}$ and let $\mathbf{w} = \overrightarrow{PR} = \langle 4, 5, 1\rangle$, where R is another point in space.

   i. Find the distance between points Q and R.

   ii. Find the projection of $\mathbf{w}$ onto $\mathbf{v}$.

   iii. Find a unit vector orthogonal to $\mathbf{v}$ and $\mathbf{w}$.

   Solution:

   (a) The points (x, y, z) equidistant from P and $z = 7$ must satisfy

   $$\boxed{\sqrt{x^2 + (y-9)^2 + (z-2)^2} = |z-7|}.$$

   (b) i. The vector $\overrightarrow{PR} = \langle 4, 5, 1\rangle$ extends from $P(0, 9, 2)$ to $R = (4, 14, 3)$. The distance between $Q(-2, 10, 5)$ and R is

   $$\sqrt{(4-(-2))^2 + (14-10)^2 + (3-5)^2} = \sqrt{6^2 + 4^2 + 2^2} = \boxed{\sqrt{56}} = 2\sqrt{14}.$$

   ii. The vector $\mathbf{v}$ equals $\langle -2, 1, 3\rangle$.

   $$\text{proj}_{\mathbf{v}}\mathbf{w} = \frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{v}|^2}\,\mathbf{v} = \frac{\langle -2, 1, 3\rangle \cdot \langle 4, 5, 1\rangle}{2^2 + 1^2 + 3^2}\langle -2, 1, 3\rangle = \boxed{\mathbf{0}}.$$

   The vectors $\mathbf{v}$ and $\mathbf{w}$ are orthogonal, so the projection is the zero vector.

   iii. The cross product of the two vectors is orthogonal to them. Let

   $$\mathbf{n} = \mathbf{v}\times\mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 3 \\ 4 & 5 & 1 \end{vmatrix} = -14\mathbf{i} + 14\mathbf{j} - 14\mathbf{k} = 14\left(-\mathbf{i} + \mathbf{j} - \mathbf{k}\right).$$

   A unit vector in the direction of $\mathbf{n}$ is $\boxed{\dfrac{-\mathbf{i}+\mathbf{j}-\mathbf{k}}{\sqrt{3}}}$ or $\boxed{\dfrac{\mathbf{i}-\mathbf{j}+\mathbf{k}}{\sqrt{3}}}$.

2. (28 pts) Let L_1 and L_2 be the lines whose symmetric equations are

   $$L_1:\ x = \frac{y+4}{2} = \frac{z-1}{-2} \qquad L_2:\ \frac{x}{-2} = \frac{y+4}{3} = \frac{z-1}{6}.$$

   (a) Write parametric equations for L_1 and L_2.

   (b) Find the point where L_1 intersects the xz plane.

   (c) Find the angle formed by L_1 and L_2.

   (d) Find an equation for the plane that contains L_1 and L_2.

   Solution:

(a) A parametric representation for L_1 is $\boxed{x = t,\ \ y = -4 + 2t,\ \ z = 1 - 2t}$. A parametric representation for L_2 is $\boxed{x = -2t,\ \ y = -4 + 3t,\ \ z = 1 + 6t}$.

(b) The line intersects the xz plane where $y = 0$ and $t = 2$. Therefore the intersection point is $\boxed{(2, 0, -3)}$.

(c) The angle is

$$\theta = \cos^{-1}\left(\frac{\mathbf{v_1} \cdot \mathbf{v_2}}{|\mathbf{v_1}|\,|\mathbf{v_2}|}\right)$$

where $\mathbf{v_1} = \langle 1, 2, -2\rangle$ and $\mathbf{v_2} = \langle -2, 3, 6\rangle$ are the direction vectors of L_1 and L_2, respectively.

$$\frac{\mathbf{v_1} \cdot \mathbf{v_2}}{|\mathbf{v_1}|\,|\mathbf{v_2}|} = \frac{\langle 1, 2, -2\rangle \cdot \langle -2, 3, 6\rangle}{\sqrt{1^2 + 2^2 + 2^2}\,\sqrt{2^2 + 3^2 + 6^2}} = \frac{-8}{3 \cdot 7} = -\frac{8}{21},$$

so the angle is $\theta = \boxed{\cos^{-1}\left(-\frac{8}{21}\right)}$.

(d) A vector normal to the plane is

$$\mathbf{n} = \mathbf{v_1} \times \mathbf{v_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -2 \\ -2 & 3 & 6 \end{vmatrix} = 18\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}.$$

The point $(0, -4, 1)$ lies on both lines, so an equation of the plane is

$$18x - 2(y + 4) + 7(z - 1) = 0 \implies \boxed{18x - 2y + 7z = 15}.$$

3. (24 pts) Consider the surface $x^2 + y^2 - z^2 - 2x + 6y - 6 = 0$.

(a) Write the equation in standard form.

(b) Identify the surface.

(c) Sketch the $z = 3$ trace.

(d) Suppose the surface is intersected with the surface $z - y = 3$. Find vector equation(s) for the curve(s) of intersection.

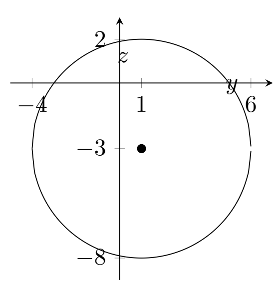
Solution:

(a)

$$\begin{aligned} x^2 + y^2 - z^2 - 2x + 6y - 6 &= 0 \\ x^2 - 2x + y^2 + 6y - z^2 &= 6 \\ \boxed{(x-1)^2 + (y+3)^2 - z^2 = 16} \end{aligned}$$

(b) The surface is a $\boxed{\text{hyperboloid of one sheet}}$.

(c) The $z = 3$ trace corresponds to $(x - 1)^2 + (y + 3)^2 = 25$ which is a circle of radius 5 centered at $(1, -3)$.



(d) Let $y = t$ and $z = y + 3 = t + 3$. Substitute into the hyperboloid equation to find an expression for x.

$$(x-1)^2 + (y+3)^2 - z^2 = (x-1)^2 + \cancel{(t+3)^2} - \cancel{(t+3)^2} = 16 \Rightarrow x = -3, 5.$$

Thus the two curves are lines with equations $\boxed{\mathbf{r_1}(t) = \langle -3, t, t+3 \rangle}$ and $\boxed{\mathbf{r_2}(t) = \langle 5, t, t+3 \rangle}$.

4. (24 pts) A bug is traveling along a path. Its position at time t seconds is $\mathbf{r}(t) = t\,\mathbf{i} + 2t\,\mathbf{j} + \frac{2}{3}t^{3/2}\,\mathbf{k}$, measured in centimeters.

(a) How far does the bug travel from $t = 0$ to 4 seconds? You may leave the final answer unsimplified.

(b) Consider the plane $4x - y - z = 13$. Is the vector tangent to the path at $t = 4$ parallel to the plane, orthogonal to the plane, or neither?

(c) At $t = 4$, the bug leaves the path and travels in a straight line in the direction of the tangent vector. Find a vector function represention $\mathbf{s}(t)$ for this straight path.

Solution:

(a) $\mathbf{r}'(t) = \mathbf{i} + 2\mathbf{j} + t^{1/2}\mathbf{k}$. The distance traveled is

$$L = \int_0^4 \left|\mathbf{r}'(t)\right|\, dt = \int_0^4 \sqrt{1^2 + 2^2 + \left(t^{1/2}\right)^2}\, dt = \int_0^4 \sqrt{5+t}\, dt$$
$$= \left[\tfrac{2}{3}(5+t)^{3/2}\right]_0^4 = \boxed{\tfrac{2}{3}\left(27 - 5^{3/2}\right)\ \text{cm}} = 18 - \frac{10\sqrt{5}}{3}\ \text{cm}.$$

(b) The tangent vector at $t = 4$ is $\mathbf{r}'(4) = \langle 1, 2, 2 \rangle$. A normal vector of the plane is $\mathbf{n} = \langle 4, -1, -1 \rangle$. These two vectors are orthogonal because their dot product is zero. Because the tangent vector and the plane are orthogonal to the same vector, they are $\boxed{\text{parallel}}$.

(c) The bug leaves the path at $\mathbf{r}(4) = 4\mathbf{i} + 8\mathbf{j} + \frac{16}{3}\mathbf{k}$ and travels in the direction of $\mathbf{r}'(4) = \langle 1, 2, 2 \rangle$. Thus

$$\mathbf{s}(t) = \boxed{\langle 4, 8, \tfrac{16}{3} \rangle + t\langle 1, 2, 2 \rangle} = \boxed{(4+t)\mathbf{i} + (8+2t)\mathbf{j} + \left(\tfrac{16}{3} + 2t\right)\mathbf{k}},\quad t \geq 0.$$