1. (40 pts) Let $g(x, y) = x^3 - 3xy + y^3$.

   (a) Find and classify the critical points of $g(x, y)$.

   (b) Find the maximum rate of change of $g(x, y)$ at the point $(2, 1)$ and the direction in which it occurs.

   (c) The origin and the point $(2, 1, 3)$ lie on the surface $z = g(x, y)$. Find an equation for the plane that passes through the points and contains the line with symmetric equations $x = \frac{y}{3} = z$.

   (d) Starting at the origin, a fly takes off from the surface $z = g(x, y)$ and travels along the path $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + 7t^2\mathbf{k}$, $t \geq 0$. At what value(s) of t will the fly meet the surface again?

Solution:

(a)

$$\begin{aligned} g(x, y) &= x^3 - 3xy + y^3 \\ g_x &= 3x^2 - 3y \\ g_y &= -3x + 3y^2 \end{aligned}$$

The critical points occur where $g_x = 0$ and $g_y = 0$.

$$\begin{aligned} g_x &= 3x^2 - 3y = 0 \implies y = x^2 \\ g_y &= -3x + 3y^2 = 0 \implies -3x + 3x^4 = 0 \implies x = 0, 1 \end{aligned}$$

There are two critical points at $(0, 0)$ and $(1, 1)$. Apply the Second Derivative Test.

$$g_{xx} = 6x \qquad g_{yy} = 6y \qquad g_{xy} = -3$$

$$\begin{aligned} &D(x, y) = g_{xx}g_{yy} - (g_{xy})^2 \\ &D(0, 0) = 0 \cdot 0 - (-3)^2 = -9 < 0 \\ &D(1, 1) = 6 \cdot 6 - (-3)^2 = 27 > 0 \quad \text{and} \quad g_{xx}(1, 1) = 6 > 0 \end{aligned}$$

Therefore there is a saddle point at $g(0, 0) = 0$ and a local minimum at $g(1, 1) = -1$.

(b)

$$\nabla g(x, y) = \langle 3x^2 - 3y, -3x + 3y^2 \rangle$$

The gradient vector $\nabla g(2, 1) = \langle 9, -3 \rangle$ is the direction of maximum rate of change, and the maximum rate is

$$|\nabla g(2, 1)| = \sqrt{9^2 + 3^2} = \sqrt{90} = 3\sqrt{10}.$$

(c) Let $v_1 = \langle 2, 1, 3 \rangle$ be the vector connecting the two points and let $v_2 = \langle 1, 3, 1 \rangle$ be the direction vector of the line. Then a normal vector to the plane is

$$v_1 \times v_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 3 \\ 1 & 3 & 1 \end{vmatrix} = -8\mathbf{i} + \mathbf{j} + 5\mathbf{k}$$

and an equation of the plane is $-8x + y + 5z = 0$.

(d) Substituting $x = t$, $y = t$, and $z = 7t^2$ into $z = g(x, y)$ gives

$$7t^2 = t^3 - 3t^2 + t^3 \implies 10t^2 = 2t^3 \implies t = 0, 5.$$

The fly begins on the surface at $t = 0$ and meets the surface again at $t = 5$.

2. (15 pts) Consider the integral

$$\int_0^3 \int_{1-x}^{1+x} \frac{x-y}{x+y}\, dy\, dx.$$

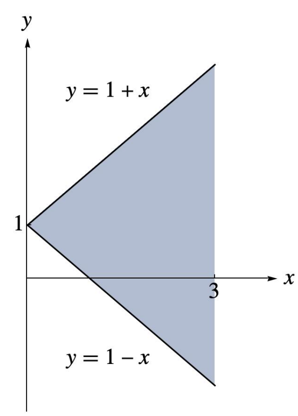
Use the transformation $u = x - y$, $v = x + y$ to set up an equivalent integral over a region in the uv plane. Sketch both the xy and uv regions. Do not evaluate the integral.

Solution:

Letting $u = x - y$ and $v = x + y$ gives $x = \frac{u+v}{2}$ and $y = \frac{v-u}{2}$. The Jacobian is

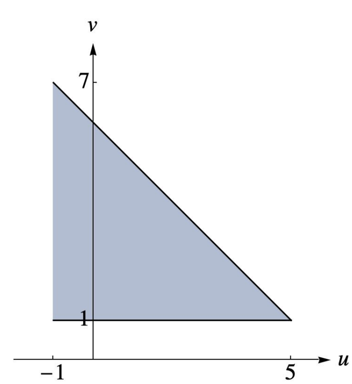
$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \tfrac{1}{2}.$$

The xy region is a triangle bounded by $y = 1 - x$, $y = 1 + x$, and $x = 3$.



xy boundaries	uv boundaries	uv equations
$y = 1 - x$	$\frac{v-u}{2} = 1 - \cdot \frac{u+v}{2}$	$v = 1$
$y = 1 + x$	$\frac{v-u}{2} = 1 + \cdot \frac{u+v}{2}$	$u = -1$
$x = 3$	$\frac{u+v}{2} = 3$	$v = 6 - u$

The corresponding uv region is a triangle bounded by $u = -1$, $v = 1$, and $u + v = 6$.



An equivalent integral over the uv-plane is

$$\int_{-1}^{5}\int_{1}^{6-u}\frac{1}{2}\frac{u}{v}\,dv\,du \qquad \text{or} \qquad \int_{1}^{7}\int_{-1}^{6-v}\frac{1}{2}\frac{u}{v}\,du\,dv.$$

3. (25 pts) The volume of a solid is given in cylindrical coordinates by $\displaystyle\int_{\pi/2}^{\pi}\int_{0}^{6}\int_{r}^{6} r\,dz\,dr\,d\theta$.

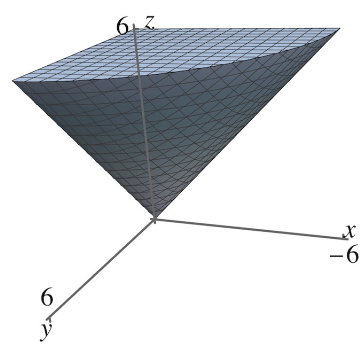
   (a) Sketch and shade the 2D cross-sections of the solid in the rz-plane (for a constant θ) and in the xy-plane. Label all intercepts.

   (b) Set up (but do not evaluate) an equivalent integral in rectangular coordinates in the order $dz\,dy\,dx$.

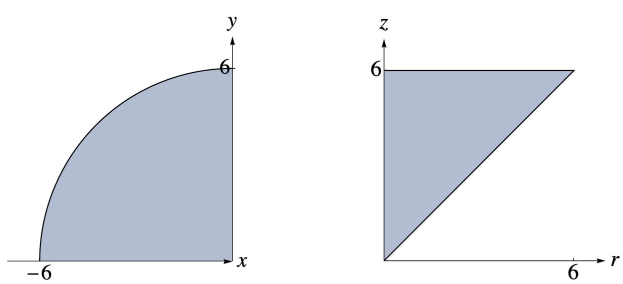
   (c) Set up (but do not evaluate) an equivalent integral in spherical coordinates in the order $d\rho\,d\phi\,d\theta$.

Solution:

The solid is a quarter cone above the second quadrant of the xy-plane, bounded below by $z = r = \sqrt{x^2+y^2}$ and above by the plane $z = 6$.



(a)



(b) In rectangular coordinates, an equation for the cone is $z = \sqrt{x^2+y^2}$. A semicircle of radius 6 centered at the origin has the equation $y = \sqrt{36-x^2}$. Therefore an equivalent integral in rectangular coordinates is

$$\int_{-6}^{0}\int_{0}^{\sqrt{36-x^2}}\int_{\sqrt{x^2+y^2}}^{6} dz\,dy\,dx.$$

(c) In spherical coordinates, the bounds for the angle θ are again $\frac{\pi}{2}$ to π. The lower and upper bounds for the angle ϕ correspond to the positive z-axis and the cone $z = r$, respectively, so $0 \le \phi \le \frac{\pi}{4}$. The lower bound for ρ is 0 at the origin, and the upper bound corresponds to

$$z = 6 \implies \rho\cos\phi = 6 \implies \rho = 6\sec\phi.$$

Therefore an equivalent integral in spherical coordinates is $\displaystyle\int_{\pi/2}^{\pi}\int_{0}^{\pi/4}\int_{0}^{6\sec\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$.

4. (25 pts)

(a) Use Gaussian elimination to solve the linear system.

$$\begin{aligned} 2x + 4y \quad\ &= -10 \\ x - 4y + z &= \ \ \ 6 \\ x + \ \ y \quad\ &= \ -4 \end{aligned}$$

(b) Reduce this homogeneous system to RREF and use the result to find the complete solution set.

$$\begin{aligned} 2x + 4y \quad\ &= 0 \\ x - 4y + z &= 0 \end{aligned}$$

Solution:

(a) First row reduce the augmented matrix $[\,\mathbf{A} \mid \mathbf{b}\,]$.

$$\left[\begin{array}{rrr|r} 2 & 4 & 0 & -10 \\ 1 & -4 & 1 & 6 \\ 1 & 1 & 0 & -4 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 2 & 0 & -5 \\ 0 & -6 & 1 & 11 \\ 0 & -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 2 & 0 & -5 \\ 0 & 1 & -\frac{1}{6} & -\frac{11}{6} \\ 0 & 0 & 1 & 5 \end{array}\right]$$

The reduced system corresponds to these equations.

$$\begin{aligned} x + 2y \qquad &= \ -5 \\ y - \tfrac{1}{6}z &= -\tfrac{11}{6} \\ z &= \ \ \ 5 \end{aligned}$$

Then use back-substitution to find $z = 5$, $y = -1$, and $x = -3$.

(b) The augmented matrix for this homogeneous system reduces to

$$\left[\begin{array}{rrr|r} 2 & 4 & 0 & 0 \\ 1 & -4 & 1 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & \frac{1}{3} & 0 \\ 0 & 1 & -\frac{1}{6} & 0 \end{array}\right].$$

The third column is a nonpivot colum, so let $z = c$. Then $x = -\frac{1}{3}c$ and $y = \frac{1}{6}c$. The complete solution set is therefore

$$\{(-\tfrac{1}{3}c, \tfrac{1}{6}c, c) \mid c \in \mathbb{R}\}.$$

The solution can be written as $c(-\frac{1}{3}, \frac{1}{6}, 1)$ or $c(-2, 1, 6)$.

5. (15 pts) Solve the linear system by finding the inverse of the coefficient matrix.

$$\begin{aligned} 3x \qquad + 3z &= 2 \\ -x - y \qquad &= 1 \\ x + y + \ \ z &= 0 \end{aligned}$$

Solution:

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 3 \\ -1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Find $\mathbf{A}^{-1}$ by row reducing the augmented matrix $[\,\mathbf{A} \mid \mathbf{I}\,]$.

$$\left[\begin{array}{rrr|rrr} 3 & 0 & 3 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & 0 & 1 & \frac{1}{3} & 0 & 0 \\ 0 & -1 & 1 & \frac{1}{3} & 1 & 0 \\ 0 & 1 & 0 & -\frac{1}{3} & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & 0 & 1 & \frac{1}{3} & 0 & 0 \\ 0 & 1 & -1 & -\frac{1}{3} & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & -1 & -1 \\ 0 & 1 & 0 & -\frac{1}{3} & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

Then multiply $\mathbf{A}^{-1}$ by $\mathbf{b}$.

$$\mathbf{A}^{-1}\mathbf{b} = \begin{bmatrix} \frac{1}{3} & -1 & -1 \\ -\frac{1}{3} & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ -\frac{2}{3} \\ 1 \end{bmatrix}$$

The solution is $x = -\frac{1}{3}$, $y = -\frac{2}{3}$, and $z = 1$.

6. (15 pts) Consider this linear system in variables x and y. For each of the following results, find nonzero coefficients a, b, c, d, e, and f. (There are multiple possible answers.)

$$\begin{aligned} ax + by &= 10 \\ cx + dy &= 5 \\ ex + fy &= 1 \end{aligned}$$

(a) The system has no solutions.

(b) The system has infinitely many solutions.

(c) The system has a unique solution.

Solution:

Here are possible answers.

(a) These equations represent parallel lines that do not intersect, so there are no solutions.

$$\begin{aligned} x + y &= 10 \\ x + y &= 5 \\ x + y &= 1 \end{aligned}$$

(b) These equations represent the same line, so there are infinitely many solutions.

$$\begin{aligned} 10x + 10y &= 10 \\ 5x + 5y &= 5 \\ x + y &= 1 \end{aligned}$$

(c) Here the first two equations represent the same line and the third equation represents a different line, so there is a unique solution.

$$\begin{aligned} 2x + 2y &= 10 \\ x + y &= 5 \\ x + 2y &= 1 \end{aligned}$$

For this alternate answer, the first equation is a combination of the other two equations representing distinct lines. The first equation equals the second equation plus 5 times the third equation. Again there will be a unique solution.

$$\begin{aligned} 6x + 11y &= 10 \\ x + y &= 5 \\ x + 2y &= 1 \end{aligned}$$

7. (15 pts) Find all square roots of the matrix A. Justify your answers. (Matrix B is a square root of A if $B^2 = A$.)

$$A = \begin{bmatrix} 1 & 10 \\ 0 & 1 \end{bmatrix}$$

Solution: Let $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $B^2 = A$ implies

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix} = \begin{bmatrix} 1 & 10 \\ 0 & 1 \end{bmatrix}$$

which leads to the system of equations

$$a^2 + bc = 1 \tag{1}$$
$$ab + bd = 10 \tag{2}$$
$$ac + cd = 0 \tag{3}$$
$$bc + d^2 = 1. \tag{4}$$

In equation (3) we have $c(a + d) = 0$ which implies that $c = 0$ or $a + d = 0$. Because $a + d = 0$ leads to a contradiction in equation (2), we have $c = 0$. Then equations (1) and (4) simplify to $a^2 = 1$ and $d^2 = 1$, respectively. We have determined that $d \neq -a$, so the possible values for a and d are $a = d = 1$ or $a = d = -1$. Substituting into equation (2) gives $b = \frac{10}{a+d} = 5$ or -5, respectively.

Thus the constants are $(a, b, c, d) = (1, 5, 0, 1)$ or $(-1, -5, 0, -1)$. The two square roots of A are

$$\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} -1 & -5 \\ 0 & -1 \end{bmatrix}.$$