1. (10 pts) Given vectors $\mathbf{a}$ and $\mathbf{b}$, prove that if

$$(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) = |\mathbf{a}|^2 + |\mathbf{b}|^2,$$

then $\mathbf{a}$ and $\mathbf{b}$ are orthogonal.

Solution:

By the distributive and commutative properties of the dot product and the identity $\mathbf{v}\cdot\mathbf{v} = |\mathbf{v}|^2$, we find that

$$\begin{aligned}(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) &= \mathbf{a}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{b} \\ &= \mathbf{a}\cdot\mathbf{a} + 2\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{b} \\ &= |\mathbf{a}|^2 + 2\mathbf{a}\cdot\mathbf{b} + |\mathbf{b}|^2.\end{aligned}$$

Substituting $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) = |\mathbf{a}|^2 + |\mathbf{b}|^2$ gives

$$|\mathbf{a}|^2 + |\mathbf{b}|^2 = |\mathbf{a}|^2 + 2\mathbf{a}\cdot\mathbf{b} + |\mathbf{b}|^2$$

and

$$0 = \mathbf{a}\cdot\mathbf{b}.$$

Therefore the vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal.

2. (12 pts) Evaluate $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy$ by converting to polar coordinates.

Solution: The xy-plane corresponds to $0 \le \theta \le 2\pi$ and $r \ge 0$ in polar coordinates. We apply the identity $r^2 = x^2 + y^2$ and insert the Jacobian r to obtain

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy = \int_0^{2\pi}\int_0^{\infty} re^{-r^2}\,dr\,d\theta.$$

The double integral can be written as the product of two single integrals.

$$\begin{aligned}&= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\infty} re^{-r^2}\,dr\right) \\ &= 2\pi\int_0^{\infty} re^{-r^2}\,dr\,d\theta\end{aligned}$$

Let $u = -r^2$, $du = -2r\,dr$.

$$\begin{aligned}&= 2\pi\int_0^{-\infty} -\tfrac{1}{2}e^u\,du \\ &= 2\pi\left(\lim_{t\to-\infty}\int_0^{t} -\tfrac{1}{2}e^u\,du\right) \\ &= 2\pi\left(\lim_{t\to-\infty} -\tfrac{1}{2}\left[e^u\right]_0^t\right) \\ &= -\pi\left(\lim_{t\to-\infty}\left(e^t-1\right)\right) = \pi\end{aligned}$$

3. (28 pts) Consider the 3D region bounded by $x = y^2$, $z = 0$, and $x + 2z = 8$.

   (a) Sketch and shade the projection of the region onto the xz-plane. Label all intercepts.

   (b) Sketch and shade the projection of the region onto the yz-plane. Label all intercepts.

   (c) Set up (but do not evaluate) triple integral(s) to find the volume of the region using rectangular coordinates in the order:

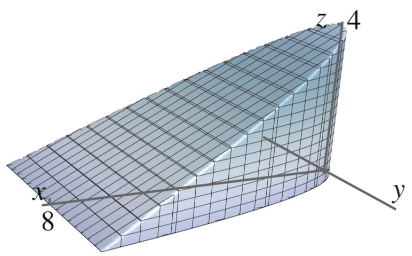
      i. $dz\,dx\,dy$

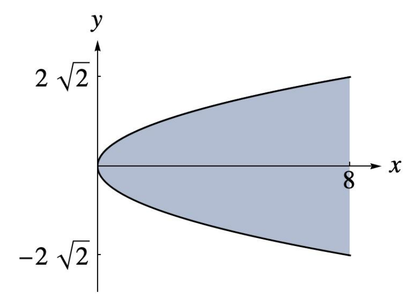
      ii. $dy\,dx\,dz$

Solution:

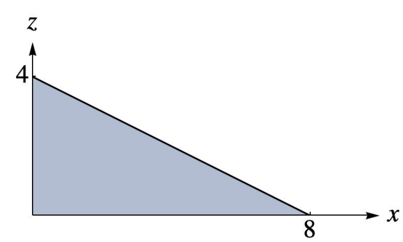
The solid is bounded by the parabolic cylinder $x = y^2$ and the planes $z = 0$ and $x+2z = 8$, as shown below on the left.

The two planes intersect at $x = 8$. The projection of the solid onto the xy plane corresponds to the region bounded by $x = y^2$ from $x = 0$ to $x = 8$, as shown below on the right.

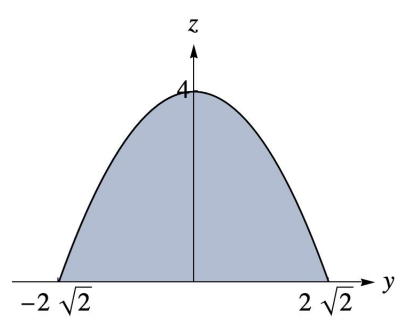




(a) The projection of the solid onto the xz plane corresponds to the region bounded by $z = 0$ and the line $z = \frac{8-x}{2}$ from $x = 0$ to $x = 8$. When $x = 0$, the z-intercept is $z = 4$.



(b) The cylinder $x = y^2$ intersects the plane $x + 2z = 8$ at $y^2 + 2z = 8$. Thus the projection of the solid onto the yz plane corresponds to the region bounded by $z = 0$ and the curve $z = \frac{8-y^2}{2}$. When $z = 0$, the y-intercepts are $y = \pm\sqrt{8} = \pm 2\sqrt{2}$, and when $y = 0$, the z-intercept is $z = 4$.



(c) i. In the xy plane, x extends from the curve $x = y^2$ to $x = 8$, and y extends from $-2\sqrt{2}$ to $2\sqrt{2}$. In the z direction, z extends from the plane $z = 0$ to the plane $z = 4 - \frac{x}{2}$. Therefore the integral is $\displaystyle\int_{-2\sqrt{2}}^{2\sqrt{2}} \int_{y^2}^{8} \int_{0}^{4-x/2} dz\,dx\,dy$.

ii. In the xz plane, x extends from 0 to the line $x = 8 - 2z$, and z extends from 0 to 4. In the y direction, y extends from one boundary of the cylinder at $y = -\sqrt{x}$ to the other boundary at $y = \sqrt{x}$. Therefore the integral is $\displaystyle\int_0^4 \int_0^{8-2z} \int_{-\sqrt{x}}^{\sqrt{x}} dy\, dx\, dz$.

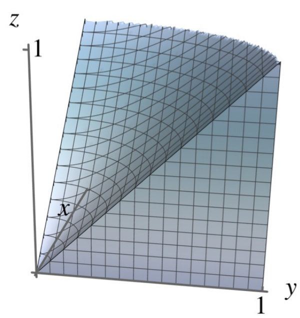
4. (20 pts) Consider the integral $\displaystyle\int_{-1}^{0} \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{x^2+y^2}} dz\, dy\, dx$. Set up (but do not evaluate) equivalent integrals using

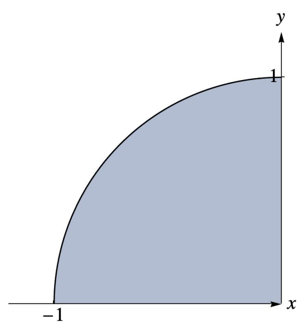
(a) cylindrical coordinates in the order $dz\, dr\, d\theta$.

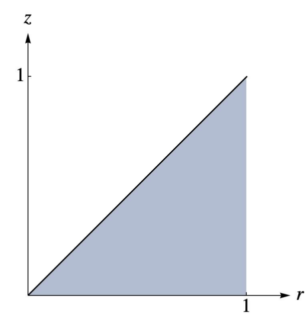
(b) spherical coordinates in the order $d\rho\, d\phi\, d\theta$.

Solution:

The integral corresponds to the volume of the solid region above the second quadrant in the xy-plane, bounded above by the cone $z = \sqrt{x^2 + y^2}$ and laterally by the circular cylinder $x^2 + y^2 = 1$.







(a) The projection of the solid onto the xy plane corresponds to the portion of a unit circle, centered at the origin, that lies in the second quadrant. It follows that $\frac{\pi}{2} \leq \theta \leq \pi$ and $0 \leq r \leq 1$.

In the z direction, the solid extends from $z = 0$ to the cone $z = \sqrt{x^2 + y^2} = \sqrt{r^2} = r$.

Therefore the integral in cylindrical coordinates is $\displaystyle\int_{\pi/2}^{\pi} \int_0^1 \int_0^r r\, dz\, dr\, d\theta$.

(b) In spherical coordinates, the bounds for the angle θ are again $\frac{\pi}{2}$ to π.

The lower and upper bounds for the angle ϕ correspond to $z = r$ and $z = 0$, respectively. If $z = r$, then

$$\rho \cos\phi = \rho \sin\phi \implies \frac{\sin\phi}{\cos\phi} = \tan\phi = 1 \implies \phi = \frac{\pi}{4},$$

and if $z = \rho\cos\phi = 0$, then $\phi = \frac{\pi}{2}$. Therefore $\frac{\pi}{4} \leq \phi \leq \frac{\pi}{2}$.

The lower bound for ρ is 0 at the origin, and the upper bound corresponds to

$$r = 1 \implies \rho \sin\phi = 1 \implies \rho = \csc\phi.$$

Therefore the integral in spherical coordinates is $\displaystyle\int_{\pi/2}^{\pi} \int_{\pi/4}^{\pi/2} \int_0^{\csc\phi} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$.

5. (15 pts) Consider the integral

$$\int_0^1 \int_{-2y}^{2y} (x-2y)\sqrt{x+2y}\,dx\,dy.$$

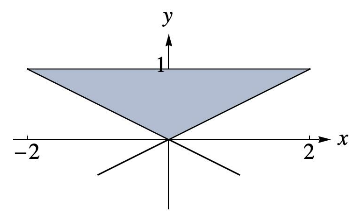
Use the transformation $u = x - 2y$, $v = x + 2y$ to set up an equivalent integral over a region in the uv-plane. Sketch both the xy and uv regions. Do not evaluate the integral.

Solution:

Letting $u = x - 2y$ and $v = x + 2y$ gives $x = \frac{u+v}{2}$ and $y = \frac{v-u}{4}$. The Jacobian is

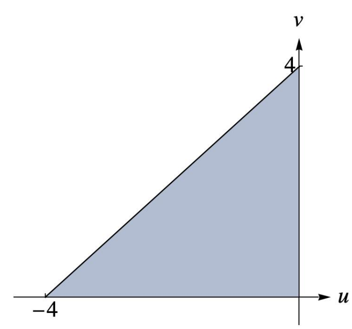
$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{4} & \frac{1}{4} \end{vmatrix} = \frac{1}{4}.$$

The xy region is a triangle bounded by $x = -2y$, $x = 2y$, and $y = 1$.



xy boundaries	uv boundaries	uv equations
$x = 2y$	$\frac{u+v}{2} = 2 \cdot \frac{v-u}{4}$	$u = 0$
$x = -2y$	$\frac{u+v}{2} = -2 \cdot \frac{v-u}{4}$	$v = 0$
$y = 1$	$\frac{v-u}{4} = 1$	$v = u + 4$

The corresponding uv region is a triangle bounded by $u = 0$, $v = 0$, and $v = u + 4$.



An equivalent integral over the uv-plane is

$$\int_{-4}^{0} \int_{0}^{u+4} \frac{1}{4} u\sqrt{v}\,dv\,du.$$

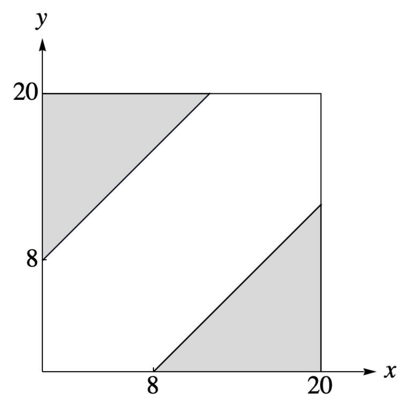
6. (15 pts) Libra and Leo decide to meet at Fiske Planetarium to see a show. Suppose each independently arrives at a time uniformly distributed between 6:40 and 7:00 pm. Evaluate a double integral to find the probability that the first to arrive has to wait longer than 8 minutes.

Solution: Let X and Y be independent random variables representing the time past 6:40 pm that Libra and Leo arrive, respectively. Each variable X and Y is uniformly distributed over $(0, 20)$. The joint probability density function of X and Y is

$$f(x,y) = \begin{cases} \frac{1}{400} & \text{if } 0 < x < 20,\ 0 < y < 20 \\ 0 & \text{otherwise.} \end{cases}$$

The desired probability can be calculated using symmetry as follows:

$$
\begin{aligned}
P(|X-Y|>8) &= 2P(X-Y>8) \\
&= 2\int_{8}^{20}\int_{0}^{x-8} \frac{1}{400}\,dy\,dx \\
&= 2\int_{8}^{20} \frac{1}{400}(x-8)\,dx \\
&= \frac{1}{200}\left[\frac{x^2}{2}-8x\right]_8^{20} \\
&= \frac{1}{200}\left((200-160)-(32-64)\right) = \frac{72}{200} = \frac{9}{25}.
\end{aligned}
$$



Alternate Solution:

$$P(|X-Y|>8) = 2P(Y-X>8) = 2\int_{0}^{12}\int_{x+8}^{20} \frac{1}{400}\,dy\,dx.$$