1. (31 pts) Let $g(x,y) = 2x^2 + 3y^2$.

   (a) i. Find a linear approximation for $g(x,y)$ centered at $(1,-1)$. You may leave your answer unsimplified.

   ii. Use the linear approximation to estimate the value of $g(1.2,-1.2)$.

   iii. Find an error bound for the estimate calculated in part ii.

1. (continued) Let $g(x,y) = 2x^2 + 3y^2$.

   (b) Find an equation for the plane tangent to the surface $z = g(x,y)$ at $x = 1, y = -1$. Simplify your answer.

   (c) Let $h(x,y) = \dfrac{xy}{g(x,y)}$. Show that $\lim\limits_{(x,y)\to(0,0)} h(x,y)$ does not exist.

Solution:

(a) i.

$$
\begin{aligned}
g(x,y) &= 2x^2 + 3y^2 \implies g(1,-1) = 5 \\
\nabla g(x,y) &= \langle 4x, 6y\rangle \implies \nabla g(1,-1) = \langle 4, -6\rangle \\
L(x,y) &= g(a,b) + g_x(a,b)(x-a) + g_y(a,b)(y-b) \\
&= g(1,-1) + g_x(1,-1)(x-1) + g_y(1,-1)(y+1) \\
&= 5 + 4(x-1) - 6(y+1) \\
&= 4x - 6y - 5
\end{aligned}
$$

ii.

$$g(1.2,-1.2) \approx L(1.2,-1.2) = 5 + 4(0.2) - 6(-0.2) = 7$$

iii. Note that $g_{xx} = 4,\ g_{yy} = 6$, and $g_{xy} = 0$. An upper bound for the second order partial derivatives is $M = 6$. Therefore an error bound for the estimate is

$$
\begin{aligned}
|E(x,y)| &\le \tfrac{1}{2!} M\left(|x-a| + |y-b|\right)^2 \\
&\le \tfrac{6}{2}\left(|1.2-1| + |-1.2+1|\right)^2 \\
&= 3\left(0.2+0.2\right)^2 = 0.48.
\end{aligned}
$$

(b) The tangent plane equation corresponds to the function $L(x,y)$.

$$z = 5 + 4(x-1) - 6(y+1) \quad \text{or} \quad z = 4x - 6y - 5.$$

(c) We wish to show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{2x^2+3y^2}$ does not exist.

Approaching $(0,0)$ along the x-axis,

$$\lim_{(x,0)\to(0,0)} \frac{xy}{2x^2+3y^2} = \lim_{(x,0)\to(0,0)} \frac{0}{2x^2} = 0.$$

Similarly the limit equals 0 approaching the origin along the y-axis.

Approaching $(0,0)$ along the line $y = mx$, however,

$$\lim_{(x,mx)\to(0,0)} \frac{xy}{2x^2+3y^2} = \lim_{(x,mx)\to(0,0)} \frac{mx^2}{2x^2+3m^2x^2} = \frac{m}{2+3m^2} \neq 0 \text{ for } m \neq 0.$$

Therefore the limit does not exist.

2. (25 pts) You are hiking on Turtle Mountain. The elevation on the mountain is given by

$$z = 1000 - \frac{1}{200}x^2 - \frac{1}{100}y^2$$

where x, y, and z are measured in meters. Suppose the positive x-axis points east and the positive y-axis points north. When you reach the location $x = 60$, $y = 40$, you stop to consider four options before continuing the hike.

(a) Option 1: You hike due south (i.e., in the negative y direction). Will you start to ascend or descend? At what rate?

(b) Option 2: You hike in the northwest direction. Will you start to ascend or descend? At what rate?

(c) Option 3: You maintain the same elevation as you hike. In which direction should you head? Write your answer as a unit vector.

(d) Option 4: You go home, taking the shortest route to the base of the mountain where you will catch a shuttle. In which direction should you head? What is the initial rate of descent?

Solution:

$$\begin{aligned} z = f(x,y) &= 1000 - \tfrac{1}{200}x^2 - \tfrac{1}{100}y^2 \\ \nabla f(x,y) &= \langle -\tfrac{1}{100}x, -\tfrac{1}{50}y \rangle \\ \nabla f(60,40) &= \langle -\tfrac{3}{5}, -\tfrac{4}{5} \rangle \end{aligned}$$

(a) Due south corresponds to the unit vector $\mathbf{u} = -\mathbf{j}$.

$$D_u f(60,40) = \nabla f(60,40) \cdot \mathbf{u} = \langle -\tfrac{3}{5}, -\tfrac{4}{5} \rangle \cdot \langle 0, -1 \rangle = \tfrac{4}{5}$$

You will ascend at a rate of $\frac{4}{5}$ vertical meters per horizontal meter.

(b) The northwest direction corresponds to the vector $-\mathbf{i} + \mathbf{j}$. Dividing by its magnitude $\sqrt{2}$ gives the unit vector $\mathbf{u} = \frac{1}{\sqrt{2}}(-\mathbf{i} + \mathbf{j})$.

$$D_u f(60,40) = \nabla f(60,40) \cdot \mathbf{u} = \langle -\tfrac{3}{5}, -\tfrac{4}{5} \rangle \cdot \tfrac{1}{\sqrt{2}} \langle -1, 1 \rangle = -\tfrac{1}{5\sqrt{2}}$$

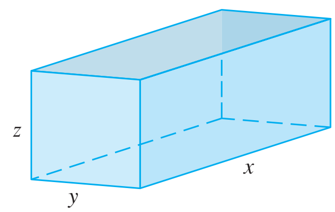
You will descend at a rate of $\frac{1}{5\sqrt{2}}$ vertical meters per horizontal meter.

(c) The elevation will remain the same if you walk along the level curve $z = f(60,40)$. The direction will be perpendicular to the gradient vector $\nabla f(60,40) = \langle -\frac{3}{5}, -\frac{4}{5} \rangle$. The two possible unit vectors are $\langle \frac{4}{5}, -\frac{3}{5} \rangle$ and $\langle -\frac{4}{5}, \frac{3}{5} \rangle$.

(d) The direction of steepest descent is $-\nabla f(60,40) = \langle \frac{3}{5}, \frac{4}{5} \rangle$. The rate of descent is $|-\nabla f(60,40)| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = 1$ vertical meter per horizontal meter.

3. (20 pts) Nemo and Dory wish to build a backyard pool for their prized collection of fish. The rectangular pool will hold 2000 cubic feet of water. The sides of the pool, to be lined with decorative tile, will cost twice as much per square foot as the concrete bottom. Use Lagrange multipliers to find the pool dimensions that will minimize the cost of construction.

Solution:



Let x and y be the dimensions in feet of the bottom of the pool, and z be the height of the pool, with $x, y, z > 0$. The volume of the pool will be xyz. The base of the pool will have an area of xy square feet, and the sides, which cost twice as much, will have an area of $2xz + 2yz$ square feet. We wish to minimize the cost $f(x, y, z) = xy + 2(2xz + 2yz)$ dollars given the constraint $g(x, y, z) = xyz = 2000$ cubic feet.

$$\begin{aligned} f(x,y,z) &= xy + 4xz + 4yz \\ \nabla f &= \langle y + 4z, x + 4z, 4x + 4y \rangle \\ g(x,y,z) &= xyz \\ \nabla g &= \langle yz, xz, xy \rangle \end{aligned}$$

$\nabla f = \lambda \nabla g$ implies

$$\begin{aligned} \lambda yz = y + 4z &\implies \lambda = \frac{y+4z}{yz} \\ \lambda xz = x + 4z &\implies \lambda = \frac{x+4z}{xz} \\ \lambda xy = 4x + 4y &\implies \lambda = \frac{4x+4y}{xy}. \end{aligned}$$

Setting the first two λ expressions equal yields

$$\frac{y+4z}{yz} = \frac{x+4z}{xz} \implies xyz + 4xz^2 = xyz + 4yz^2 \implies x = y \text{ because } z \neq 0.$$

Setting the second two λ expressions equal and substituting $x = y$ yields

$$\frac{x+4z}{xz} = \frac{4x+4x}{x^2} = \frac{8}{x} \implies x^2 + 4xz = 8xz \implies x^2 = 4xz \implies z = \tfrac{x}{4} \text{ because } x \neq 0.$$

Substituting $x = y$ and $z = \frac{x}{4}$ into the original constraint gives

$$xyz = x \cdot x \cdot \tfrac{x}{4} = 2000 \implies x^3 = 8000 \implies x = 20,\ y = 20,\ z = 5.$$

Therefore the optimal dimensions are 20 feet by 20 feet for the bottom of the pool and 5 feet for the height.

4. The following two problems are not related.

(a) (12 pts) Find and classify all critical points of $f(x,y) = x^2e^y + ye^y$.

(b) (12 pts) The temperature in degrees Celsius at a point (x,y,z) is given by

$$T(x,y,z) = 2x^2 - xyz.$$

A particle moving through space has the position function $\mathbf{r}(t) = 2t^2\mathbf{i} + 2t\mathbf{j} - t^2\mathbf{k}$, where time t is measured in seconds. In degrees per second, how fast is the temperature along the particle's path changing when it reaches the point $\left(\frac{1}{2}, 1, -\frac{1}{4}\right)$?

Solution:

(a)

$$\begin{aligned} f(x,y) &= x^2e^y + ye^y \\ f_x &= 2xe^y \\ f_y &= x^2e^y + ye^y + e^y = e^y\left(x^2 + y + 1\right) \end{aligned}$$

The critical points occur where $f_x = 0$ and $f_y = 0$.

$$\begin{aligned} f_x &= 0 \implies 2xe^y = 0 \implies x = 0 \\ f_y &= 0 \implies e^y(0 + y + 1) = 0 \implies y = -1 \end{aligned}$$

There is one critical point at $(0,-1)$. Apply the Second Derivative Test.

$$\begin{aligned} f_{xx} &= 2e^y & f_{xx}(0,-1) &= 2e^{-1} \\ f_{yy} &= e^y + e^y\left(x^2 + y + 1\right) & f_{yy}(0,-1) &= e^{-1} \\ f_{xy} &= 2xe^y & f_{xy}(0,-1) &= 0 \end{aligned}$$

$$\begin{aligned} D(0,-1) &= f_{xx}(0,-1)f_{yy}(0,-1) - \left(f_{xy}(0,-1)\right)^2 \\ &= 2e^{-1} \cdot e^{-1} - 0^2 = 2e^{-2} \end{aligned}$$

Because $D(0,-1) > 0$ and $f_{xx}(0,-1) > 0$, there is a local minimum at $f(0,-1) = -e^{-1}$.

(b) The rate of temperature change with respect to time t is

$$\begin{aligned} \frac{dT}{dt} &= \frac{\partial T}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial T}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial T}{\partial z} \cdot \frac{dz}{dt} \\ &= \nabla T(x,y,z) \cdot \langle \tfrac{dx}{dt}, \tfrac{dy}{dt}, \tfrac{dz}{dt} \rangle \\ &= \langle 4x - yz, -xz, -xy \rangle \cdot \langle 4t, 2, -2t \rangle. \end{aligned}$$

The point $\left(\frac{1}{2}, 1, -\frac{1}{4}\right)$ corresponds to $t = \frac{1}{2}$. At this point the rate of change is

$$\left.\frac{dT}{dt}\right|_{t=1/2} = \langle 2\tfrac{1}{4}, \tfrac{1}{8}, -\tfrac{1}{2} \rangle \cdot \langle 2, 2, -1 \rangle = 5\tfrac{1}{4} = \tfrac{21}{4}.$$

The temperature is increasing at $\frac{21}{4}$ degrees per second.