1. (30 pts) Consider the two planes described by $2y + z = 1$ and $-x + 2y + 2z = 3$.

   (a) Is the point $P(-1, 0, 1)$ on both planes, one of the planes, or neither?

   (b) Find the angle formed by the two planes.

   (c) Find equations for the line of intersection of these two planes. Express your answer in parametric and symmetric forms.

   (d) What is the shortest distance between the point $Q(0, 1, 3)$ and the line of intersection?

Solution:

(a) The point $P(-1, 0, 1)$ satisfies both equations, so it lies on both planes.

$$0 + 1 = 1 \quad \text{and} \quad 1 + 0 + 2 = 3$$

(b) The angle θ between the planes equals the angle between their normal vectors $\mathbf{n_1} = \langle 0, 2, 1\rangle$ and $\mathbf{n_2} = \langle -1, 2, 2\rangle$.

$$\cos\theta = \frac{\mathbf{n_1} \cdot \mathbf{n_2}}{|\mathbf{n_1}|\,|\mathbf{n_2}|} = \frac{\langle 0, 2, 1\rangle \cdot \langle -1, 2, 2\rangle}{\sqrt{2^2 + 1^2} \cdot \sqrt{1^2 + 2^2 + 2^2}} = \frac{6}{3\sqrt{5}} = \frac{2}{\sqrt{5}}$$

The angle is $\theta = \cos^{-1}\left(\frac{2}{\sqrt{5}}\right)$.

(c) The point $P(-1, 0, 1)$ lies on the line of intersection. The direction of the line is

$$\mathbf{v} = \mathbf{n_1} \times \mathbf{n_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{vmatrix} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}.$$

Therefore parametric equations for the line of intersection are

$$x = -1 + 2t, \;\; y = -t, \;\; z = 1 + 2t$$

and symmetric equations are

$$\frac{x+1}{2} = \frac{y}{-1} = \frac{z-1}{2}.$$

(d) The shortest distance between the point $Q(0, 1, 3)$ and the line of intersection is

$$d = \frac{|\mathbf{PQ} \times \mathbf{v}|}{|\mathbf{v}|}$$

where $\mathbf{PQ} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$ is the vector from $P(-1, 0, 1)$ to $Q(0, 1, 3)$, and $\mathbf{v} = \mathbf{n_1} \times \mathbf{n_2} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

$$\mathbf{PQ} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 2 \\ 2 & -1 & 2 \end{vmatrix} = 4\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$$

$$|\mathbf{PQ} \times \mathbf{v}| = \sqrt{4^2 + 2^2 + 3^2} = \sqrt{29},$$

thus

$$d = \frac{|\mathbf{PQ} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{29}}{\sqrt{2^2+1^2+2^2}} = \frac{\sqrt{29}}{3}.$$

Alternate Solution

The shortest distance equals

$$\begin{aligned}
|\mathbf{PQ} - \text{proj}_{\mathbf{v}}\mathbf{PQ}| &= \left|\mathbf{PQ} - \frac{\mathbf{PQ}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v}\right| \\
&= \left|\langle 1,1,2\rangle - \frac{\langle 1,1,2\rangle\cdot\langle 2,-1,2\rangle}{3^2}\langle 2,-1,2\rangle\right| \\
&= \left|\langle 1,1,2\rangle - \tfrac{5}{9}\langle 2,-1,2\rangle\right| \\
&= \left|\langle -\tfrac{1}{9}, \tfrac{14}{9}, \tfrac{8}{9}\rangle\right| = \tfrac{\sqrt{29}}{3}.
\end{aligned}$$

2. (16 pts) A particle is moving in the direction $\mathbf{v} = \mathbf{i} + \mathbf{j}$ when a force of $\mathbf{F} = 3\mathbf{j} + 4\mathbf{k}$ is applied to it.

   (a) Decompose the vector $\mathbf{F}$ into a sum of 2 vectors: one vector parallel to the particle's direction of motion and the other vector orthogonal.

   (b) Find a unit vector that is orthogonal to $\mathbf{F}$.

Solution:

(a) A vector parallel to the particle's direction of motion is the projection of $\mathbf{F}$ onto $\mathbf{v}$.

$$\text{proj}_{\mathbf{v}}\mathbf{F} = \frac{\mathbf{v}\cdot\mathbf{F}}{|\mathbf{v}|^2}\,\mathbf{v} = \frac{\langle 1,1,0\rangle\cdot\langle 0,3,4\rangle}{\left(\sqrt{2}\right)^2}\,\langle 1,1,0\rangle = \tfrac{3}{2}\langle 1,1,0\rangle = \left\langle \tfrac{3}{2}, \tfrac{3}{2}, 0\right\rangle.$$

A vector orthogonal to the projection is

$$\text{orth}_{\mathbf{v}}\mathbf{F} = \mathbf{F} - \text{proj}_{\mathbf{v}}\mathbf{F} = \langle 0,3,4\rangle - \left\langle \tfrac{3}{2}, \tfrac{3}{2}, 0\right\rangle = \left\langle -\tfrac{3}{2}, \tfrac{3}{2}, 4\right\rangle.$$

Therefore

$$\mathbf{F} = \langle 0,3,4\rangle = \left\langle \tfrac{3}{2}, \tfrac{3}{2}, 0\right\rangle + \left\langle -\tfrac{3}{2}, \tfrac{3}{2}, 4\right\rangle.$$

(b) The vector $\mathbf{F} = 3\mathbf{j} + 4\mathbf{k}$ lies in the yz-plane and has slope $\frac{4}{3}$. An orthogonal vector with slope $-\frac{3}{4}$ is $\mathbf{w} = 4\mathbf{j} - 3\mathbf{k}$. A unit vector in the direction of $\mathbf{w}$ is

$$\frac{\mathbf{w}}{|\mathbf{w}|} = \frac{1}{\sqrt{4^2+3^2}}\,(4\mathbf{j} - 3\mathbf{k}) = \tfrac{1}{5}\,(4\mathbf{j} - 3\mathbf{k}) = \tfrac{4}{5}\mathbf{j} - \tfrac{3}{5}\mathbf{k}.$$

Alternate solution

Let the orthogonal vector be $\mathbf{w} = \langle a,b,c\rangle$. Then

$$\mathbf{F}\cdot\mathbf{w} = \langle 0,3,4\rangle\cdot\langle a,b,c\rangle = 3b + 4c = 0.$$

Choose any values of b and c that satisfy the equation, for example $b = -4$ and $c = 3$. Then $\mathbf{w} = \langle 0,-4,3\rangle$ and a unit vector is

$$\frac{\mathbf{w}}{|\mathbf{w}|} = \frac{1}{\sqrt{4^2+3^2}}\langle 0,-4,3\rangle = \tfrac{1}{5}\langle 0,-4,3\rangle = \left\langle 0, -\tfrac{4}{5}, \tfrac{3}{5}\right\rangle.$$

3. (30 pts) Consider the surface $3x^2 + 6x + y + 3z^2 + 3 = 0$.

   (a) Write the equation in standard form.

   (b) For each plane listed below,

   - identify the shape of the trace of the surface in the plane, and
   - find the (x, y, z) coordinates of either the center or the vertex of the trace.

   i. $y = -2$ plane

   ii. xy-plane

   (c) Identify the given surface.

   (d) Find a vector equation for the line along which the surface is centered.

Solution:

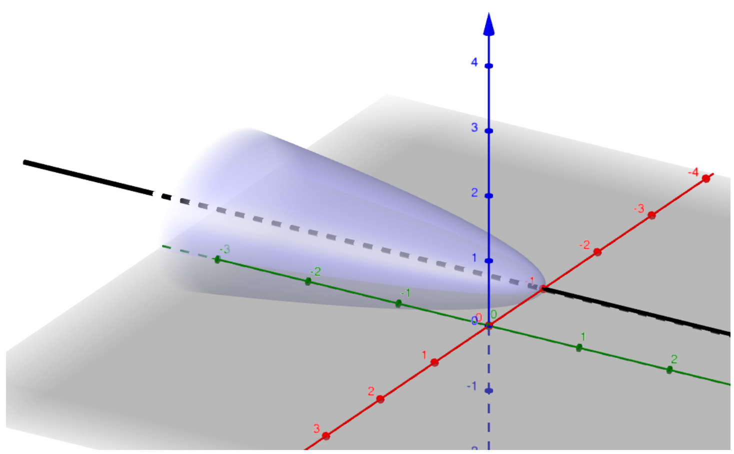
(a) $3(x+1)^2 + y + 3z^2 = 0$

(b) i. The $y = -2$ plane produces a trace with equation $3(x+1)^2 + 3z^2 = 2$. The trace is a circle with center at $(-1, -2, 0)$.

ii. The xy-plane corresponds to $z = 0$ which produces a trace with the equation $3(x+1)^2 + y = 0$. The trace is a parabola with vertex at $(-1, 0, 0)$.

(c) The surface is a circular paraboloid.

(d) The surface has a vertex at $(-1, 0, 0)$ and is centered along a line in the xy-plane parallel to the y-axis. A vector equation for the line is $\mathbf{r}(t) = \langle -1, 0, 0\rangle + t\langle 0, 1, 0\rangle = \langle -1, t, 0\rangle$.



4. (24 pts) A particle travels along the following path, starting at $t = 0$.

$$\mathbf{r}(t) = (3\cos t)\,\mathbf{i} + (3\sin t)\,\mathbf{j} + \left(\sqrt{7}\,t\right)\mathbf{k}$$

   (a) Find the velocity vector $\mathbf{v}(t)$ of the path.

   (b) Find the unit tangent vector $\mathbf{T}(t)$. Simplify your answer.

   (c) At $t = \pi$, how far is the particle from its starting position?

   (d) At $t = \pi$, how far has the particle traveled along the path?

Solution:

(a) $\mathbf{v}(t) = \mathbf{r}'(t) = (-3\sin t)\,\mathbf{i} + (3\cos t)\,\mathbf{j} + \sqrt{7}\,\mathbf{k}$

(b) The unit tangent vector is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}.$$

Note that

$$|\mathbf{v}(t)| = \sqrt{9\sin^2 t + 9\cos^2 t + 7} = \sqrt{9+7} = 4.$$

Thus

$$\begin{aligned} \mathbf{T}(t) &= \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \frac{1}{4}\Big((-3\sin t)\,\mathbf{i} + (3\cos t)\,\mathbf{j} + \sqrt{7}\,\mathbf{k}\Big) \\ &= \big(-\tfrac{3}{4}\sin t\big)\,\mathbf{i} + \big(\tfrac{3}{4}\cos t\big)\,\mathbf{j} + \tfrac{\sqrt{7}}{4}\mathbf{k}. \end{aligned}$$

(c) The displacement is

$$|\mathbf{r}(\pi) - \mathbf{r}(0)| = \Big|\big(-3\mathbf{i} + \sqrt{7}\pi\mathbf{k}\big) - 3\mathbf{i}\Big| = \Big|-6\mathbf{i} + \sqrt{7}\pi\mathbf{k}\Big| = \sqrt{36 + 7\pi^2}.$$

(d) The distance traveled along the path is

$$L = \int_0^\pi \big|\mathbf{r}'(t)\big|\;dt = \int_0^\pi |\mathbf{v}(t)|\;dt = \int_0^\pi 4\,dt = [4t]_0^\pi = 4\pi.$$