Work out the following problems, fully simplifying your answers.

1. (30 pts) Evaluate the following integrals.

$$(a)\ \int_0^{\pi/2} \sin^2(\theta)\cos^3(\theta)\,\mathrm{d}\theta \qquad (b)\ \int_2^{\infty} \frac{2}{(1+x)(1-x)}\,\mathrm{d}x$$

Solution:

(a) Using a u-sub with $u = \sin(\theta)$, $\mathrm{d}u = \cos(\theta)$, we have

$$\begin{aligned}
\int_0^{\pi/2} \sin^2(\theta)\cos^3(\theta)\,\mathrm{d}\theta &= \int_0^{\pi/2} \sin^2(\theta)\cos^2(\theta)\cos(\theta)\,\mathrm{d}\theta \\
&= \int_0^{\pi/2} \sin^2(\theta)(1-\sin^2(\theta))\cos(\theta)\,\mathrm{d}\theta \\
&= \int_0^{1} u^2(1-u^2)\,\mathrm{d}u \\
&= \frac{1}{3}u^3 - \frac{1}{5}u^5\bigg|_0^1 \\
&= \frac{1}{3} - \frac{1}{5} \\
&= \boxed{\frac{2}{15}.}
\end{aligned}$$

(b) To start, we will use partial fractions as

$$\frac{2}{(1+x)(1-x)} = \frac{A}{1+x} + \frac{B}{1-x} \implies 2 = A(1-x) + B(1+c).$$

Solving for A and B gives $A = B = 1$. Then, we have

$$\begin{aligned}
\int_2^{\infty} \frac{2}{(1+x)(1-x)}\,\mathrm{d}x &= \lim_{t\to\infty} \int_2^{t} \frac{1}{1+x} + \frac{1}{1-x}\,\mathrm{d}x \\
&= \lim_{t\to\infty} \ln|1+x| - \ln|1-x|\bigg|_2^t \\
&= \lim_{t\to\infty} \ln\left|\frac{1+x}{1-x}\right|\bigg|_2^t \\
&= \lim_{t\to\infty} \ln\left|\frac{1+t}{1-t}\right| - \ln\left|\frac{1+2}{1-2}\right| \\
&= \ln 1 - \ln 3 \\
&= \boxed{-\ln 3.}
\end{aligned}$$

2. (15 pts) Solve the following initial value problem for y as a function of x.

$$\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}x} = -2xy \\ y(0) = 2 \end{cases}$$

Solution: Using separation of variables, we obtain the general solution

$$\int \frac{1}{y}\,\mathrm{d}y = \int -2x\,\mathrm{d}x \implies \ln|y| = -x^2 + C \implies y = e^{-x^2+C} = Ae^{-x^2}$$

where $A = e^C$. Applying the initial condition, we have

$$y(0) = 2 = Ae^0 \implies A = 2.$$

Putting everything together, we have the solution $\boxed{y = 2e^{-x^2}.}$

3. (20 pts) Determine the radius and interval of convergence for the following power series.

$$\sum_{n=0}^{\infty} \frac{10^n (x-5)^n}{n!}$$

Solution: Applying the ratio test, we have

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{10^{n+1}(x-5)^{n+1}}{(n+1)!} \cdot \frac{n!}{10^n (x-5)^n}\right| \\
&= \left|\frac{10^n 10 (x-5)^n (x-5)}{(n+1)n!} \cdot \frac{n!}{10^n (x-5)^n}\right| \\
&= \left|\frac{10(x-5)}{n+1}\right| \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Since the ratio limit goes to zero for all values of x, the radius of convergence is $\boxed{R = \infty}$ and the interval of convergence is $\boxed{I = (-\infty, \infty).}$

4. (25 pts) Given the function $f(x) = \sin(x)\cos(x)$, answer the following:
   (a) Using any method you'd like, compute $T_3(x)$ for the Maclaurin series of $f(x) = \sin(x)\cos(x)$.
   (b) Assuming $\left|f^{(4)}(x)\right| \leq 8$ for all values of x, find an error bound in using $T_3(x)$ to approximate $f(-0.1)$.

Solution:

(a) **Option 1:** Using the definition of a Maclaurin series, we have

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$f^{(n)}(0)/n!$
0	$\sin(x)\cos(x)$	0	0
1	$\cos^2(x) - \sin^2(x)$	1	1
2	$-4\sin(x)\cos(x)$	0	0
3	$-4(\cos^2(x) - \sin^2(x))$	-4	$-4/3!$

From the table, we have

$$T_3(x) = 0 + x + 0x^2 - \frac{4}{3!}x^3 \implies \boxed{T_3(x) = x - \frac{2}{3}x^3.}$$

Option 2 Using series multiplication, we have

$$\begin{aligned} f(x) &= \sin(x)\cos(x) \\ &= \left(x - \frac{x^3}{3!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \cdots\right) \\ &= x - \frac{x^3}{2} - \frac{x^3}{3!} + \cdots \\ &= x - \frac{2}{3}x^3 + \cdots \end{aligned}$$

Taking the terms up to degree three, we have

$$\boxed{T_3(x) = x - \frac{2}{3}x^3.}$$

Option 3: Using a trig identity and the Maclaurin series for $\sin(x)$, we have that

$$f(x) = \sin(x)\cos(x) = \frac{1}{2}\sin(2x) = \frac{1}{2}\left((2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} + \cdots\right) = x - \frac{2}{3}x^3 + \frac{(2x)^5}{5!} + \cdots$$

Collecting terms up to degree three yields

$$\boxed{T_3(x) = x - \frac{2}{3}x^3.}$$

(b) Using the Taylor Remainder Theorem and the fact that the fourth derivative is bounded by 8, we have that the error (or remainder) is given by

$$|R_3(-0.1)| = \left|\frac{f^{(4)}(z)}{4!}(-0.1)^4\right| \leq \left|\frac{8}{4!}(-0.1)^4\right| = \boxed{\frac{8}{4! \cdot 10^4} = \frac{1}{30000}}.$$

5. (30 pts) Consider the parametric equations given below.

$$\begin{cases} x = t^2 \\ y = \sin t \end{cases}, \qquad 0 \le t \le \pi$$

Answer the following:

(a) Setup and evaluate an integral with respect to t to find the area between the curve and the x-axis.

(b) Assuming $t \ge 0$, eliminate the variable t from the parametric equations to find an equation of the curve in terms of x and y.

Solution:

(a) To find the area under the curve, we have

$$A = \int_\alpha^\beta y\,\mathrm{d}x = \int_a^b x' y\,\mathrm{d}t = \int_0^\pi 2t\sin t\,\mathrm{d}t.$$

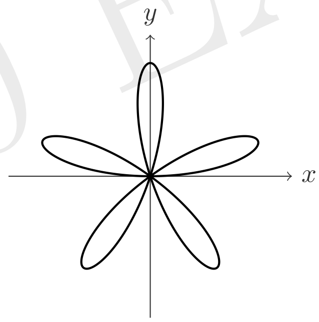
Using integration by parts with $u = 2t, \mathrm{d}u = 2\,\mathrm{d}t, \mathrm{d}v = \sin t\,\mathrm{d}t$, and $v = -\cos t$, we have

$$A = -2t\cos t\Big|_0^\pi + \int_0^\pi 2\cos t\,\mathrm{d}t = 2\pi + 2\sin t\Big|_0^\pi = \boxed{2\pi.}$$

(b) Since $x = t^2$ and $t \ge 0$, we have that $t = \sqrt{x}$. Plugging this value of t into the y equation yields

$$\boxed{y = \sin(\sqrt{x}).}$$

6. (30 pts) Consider the polar curve defined by $r = \sin(5\theta)$ for $0 \le \theta \le \pi$ (plotted below).



Answer the following:

(a) Setup, **but do not evaluate**, an integral to find the total length of the curve $r = \sin(5\theta)$.

(b) Evaluate an integral to the find the area enclosed by one petal of the curve $r = \sin(5\theta)$.

Solution:

(a) To compute arclength, we need the derivative, $r' = 5\cos(5\theta)$. Plugging into the length element, we have

$$\mathrm{d}s = \sqrt{r^2 + (r')^2}\,\mathrm{d}\theta = \sqrt{\sin^2(5\theta) + 25\cos^2(5\theta)}\,\mathrm{d}\theta.$$

Putting everything together, the arclength can be computed as

$$\boxed{L = \int_a^b \mathrm{d}s = \int_0^\pi \sqrt{\sin^2(5\theta) + 25\cos^2(5\theta)}\,\mathrm{d}\theta.}$$

(b) Before we can setup our area integral, we need to find the bounds of integration. Since each loop starts and ends at the origin, we can find the bounds of integration by solving for the values of θ that make $r = 0$. In this case, we have

$$r = \sin(5\theta) = 0 \implies 5\theta = 0, \pi, 2\pi, 3\pi, \ldots \implies \theta = 0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \ldots.$$

Taking any two consecutive angles gives us our bounds of integration. Taking the first consecutive angles, the area of a loop can be computed as

$$\begin{aligned}
A = \int_a^b \frac{1}{2} r^2\,\mathrm{d}\theta &= \frac{1}{2}\int_0^{\pi/5} \sin^2(5\theta)\,\mathrm{d}\theta \\
&= \frac{1}{2}\int_0^{\pi/5} \frac{1}{2}(1 - \cos(10\cdot\theta))\,\mathrm{d}\theta \\
&= \frac{1}{4}\left(\theta - \frac{1}{10}\sin(10\cdot\theta)\right)\Bigg|_0^{\pi/5} \\
&= \frac{1}{4}\left(\frac{\pi}{5} - \frac{1}{10}\sin(2\pi) - 0 + \frac{1}{10}\sin(0)\right) \\
&= \boxed{\frac{\pi}{20}}.
\end{aligned}$$

FORMULAS ON BACK

Trigonometric Identities

$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ $\quad \sin^2 x = \frac{1}{2}(1 - \cos 2x)$ $\quad \sin 2x = 2\sin x \cos x$ $\quad \cos 2x = \cos^2 x - \sin^2 x$

Common Maclaurin Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad R = 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad R = 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots \qquad R = 1$$