Work out the following problems and simplify your answers.

1. (30 pts) Determine if the following series converge or diverge. Fully justify your answer and state which test you used.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n+5}{\sqrt[3]{n^7+n^2}}$ (c) $\displaystyle\sum_{n=0}^{\infty} \frac{3^{n+1}n!}{(2n)!}$

Solution:

(a) First, we note that we can use the integral test since $\dfrac{1}{n(\ln n)^2}$ is positive and decreasing for $n \geq 2$. Using the integral test, we have

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\,\mathrm{d}x = \lim_{t\to\infty} \int_2^t \frac{1}{x(\ln x)^2}\,\mathrm{d}x = \lim_{t\to\infty} \int_{\ln 2}^t \frac{1}{u^2}\,\mathrm{d}u = -\lim_{t\to\infty} \frac{1}{u}\bigg|_{\ln 2}^t = \lim_{t\to\infty} \frac{1}{\ln 2} - \frac{1}{t} = \frac{1}{\ln 2}$$

showing that the integral converges. By the integral test, $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges.

(b) We can show convergence using the direct comparison test. By making the denominator smaller, we get the following useful inequality

$$0 < \frac{n+5}{\sqrt[3]{n^7+n^2}} < \frac{n+5}{\sqrt[3]{n^7}} = \frac{n}{\sqrt[3]{n^7}} + \frac{5}{\sqrt[3]{n^7}} = \frac{1}{n^{4/3}} + \frac{5}{n^{7/3}}.$$

Next, we note that

$$\sum_{n=1}^{\infty} \frac{1}{n^{4/3}} + \frac{5}{n^{7/3}} = \sum_{n=1}^{\infty} \frac{1}{n^{4/3}} + \sum_{n=1}^{\infty} \frac{5}{n^{7/3}}$$

is the sum of two convergent p-series with $p = 4/3 > 1$ and $p = 7/3 > 1$ respectively and is thus convergent itself. To finish up, by the direct comparison test $\displaystyle\sum_{n=1}^{\infty} \frac{n+5}{\sqrt[3]{n^7+n^2}}$ converges.

(c) Using the ratio test, we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{3^{n+2}(n+1)!}{(2(n+1))!} \cdot \frac{(2n)!}{3^{n+1}n!}\right| = \left|\frac{3\cdot 3^{n+1}(n+1)n!}{(2n+2)(2n+1)(2n)!} \cdot \frac{(2n)!}{3^{n+1}n!}\right| = \left|\frac{3(n+1)}{(2n+2)(2n+1)}\right|$$

and taking the limit yields

$$L = \lim_{n\to\infty} \left|\frac{3(n+1)}{(2n+2)(2n+1)}\right| = 0.$$

Since $L = 0 < 1$, the ratio test tells us that $\displaystyle\sum_{n=0}^{\infty} \frac{3^{n+1}n!}{(2n)!}$ converges.

2. (15 pts) Find the sum of the following series $\displaystyle\sum_{n=0}^{\infty} \frac{3}{n^2+3n+2}$. (Hint: use partial fractions)

Solution: We start with partial fractions as

$$\frac{3}{n^2+3n+2} = \frac{3}{(n+1)(n+2)} = \frac{A}{n+1} + \frac{B}{n+2}.$$

Solving for A and B yields $A = 3$ and $B = -3$. Assuming that we have a telescoping series, we compute the partial sums as

$$\begin{aligned} s_n = \sum_{i=0}^{n} \frac{3}{n^2+3n+2} &= \sum_{i=0}^{n} \frac{3}{n+1} - \frac{3}{n+2} \\ &= \left(\frac{3}{1} - \frac{3}{2}\right) + \left(\frac{3}{2} - \frac{3}{3}\right) + \left(\frac{3}{3} - \frac{3}{4}\right) + \cdots + \left(\frac{3}{n+1} - \frac{3}{n+2}\right) \\ &= \frac{3}{1} \cancel{- \frac{3}{2} + \frac{3}{2}} \cancel{- \frac{3}{3} + \frac{3}{3}} \cancel{- \frac{3}{4} + \cdots + \frac{3}{n+1}} - \frac{3}{n+2} \\ &= 3 - \frac{3}{n+2}. \end{aligned}$$

Then, we can compute the actual sum as

$$\boxed{\sum_{n=0}^{\infty} \frac{3}{n^2+3n+2} = \lim_{n\to\infty} s_n = \lim_{n\to\infty} 3 - \frac{3}{n+2} = 3.}$$

3. (20 pts) Consider the series $\frac{1}{10} - \frac{1}{20} + \frac{1}{30} - \frac{1}{40} \pm \cdots$.

   (a) Find a_n and write the series in sigma notation, $\displaystyle\sum_{n=1}^{\infty} a_n$.

   (b) Is the series absolutely convergent, conditionally convergent, or divergent. Justify your answer.

   (c) How many terms are needed to estimate the actual sum to an error less than 10^{-3}?

Solution:

(a) The problem has us starting at $n = 1$. With this in mind, we have that

$$\boxed{a_n = \frac{(-1)^{n-1}}{10n}} \quad \Longrightarrow \quad \boxed{\frac{1}{10} - \frac{1}{20} + \frac{1}{30} - \frac{1}{40} \pm \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{10n}.}$$

(b) We start by checking for absolute convergence as

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{10n} \right| = \frac{1}{10} \sum_{n=1}^{\infty} \frac{1}{n}$$

which is the divergent harmonic series. Since the series is not absolutely convergent, we will check the convergence of the original series using the alternating series test. Let $b_n = \frac{1}{10n}$. Clearly b_n is decreasing and $\lim_{n\to\infty} b_n = 0$. Then, by the alternating series test, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{10n}$ is convergent and hence conditionally convergent.

(c) From the previous part, we know the alternating series estimation theorem applies. So, we wish to find n such that

$$b_{n+1} = \frac{1}{10(n+1)} < 10^{-3}.$$

Rearranging the inequality yields $n + 1 > 100$ which implies we need $\boxed{n > 99.}$

4. (15 pts) Suppose $f(x)$ has a power series representation $\sum_{n=0}^{\infty} c_n (x-a)^n$ with an interval of convergence of $(0, 2]$.

   (a) Find the center and radius of convergence of the series.

   (b) Determine whether the following series are convergent, divergent, or if more information is needed to make a conclusion. Justify your answers.

$$\text{(i)} \sum_{n=0}^{\infty} c_n \frac{1}{2^{n+1}} \qquad \text{(ii)} \sum_{n=0}^{\infty} c_n 2^{3n} \qquad \text{(iii)} \sum_{n=0}^{\infty} c_n (-1)^n$$

Solution:

(a) The center is given by the halfway point from 0 to 2 which is $\boxed{a = 1.}$ The radius of convergence is half the length of the interval $\boxed{R = \frac{2-0}{2} = 1.}$

(b) We can check the convergence by figuring out what x-value was plugged into each series.

   i. The x-value used here is given by $x - 1 = \frac{1}{2}$ which implies $x = 3/2$ which is in our interval of convergence and hence converges.

   ii. We first rewrite the series as

$$\sum_{n=0}^{\infty} c_n 2^{3n} = \sum_{n=0}^{\infty} c_n 8^n.$$

   The x-value used here is given by $x - 1 = 8$ which implies $x = 9$ which is not in our interval of convergence and hence diverges

   iii. The x-value used here is given by $x - 1 = -1$ which implies $x = 0$ which is on the left endpoint of our interval of convergence which is not included in the interval. So, this series diverges.

5. (20 pts) Compute the following stating the radius of convergence for each part.

   (a) Write out the power series centered at 0 for $\dfrac{1}{1-x}$.

   (b) Find a power series centered at 0 of $\dfrac{1}{3+x}$.

   (c) By integrating, find a power series for $\ln(3+x)$.

   (d) Find the power series for $x^2\ln(3+x)$.

Solution:

(a) $\dfrac{1}{1-x}$ is our simple geometric series given by

$$\boxed{\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n, \qquad R=1.}$$

(b) Using the geometric series, we have

$$\frac{1}{3+x}=\frac{1}{3}\frac{1}{1+\frac{x}{3}}=\frac{1}{3}\frac{1}{1-\left(-\frac{x}{3}\right)}=\frac{1}{3}\sum_{n=0}^{\infty}\left(-\frac{x}{3}\right)^n=\boxed{\sum_{n=0}^{\infty}(-1)^n\frac{1}{3^{n+1}}x^n.}$$

The radius of convergence can be found by noting that we only have convergence when

$$\left|-\frac{x}{3}\right|<1 \implies |x|<3 \implies \boxed{R=3.}$$

(c) We can find $\ln(3+x)$ using term-by-term integration as

$$\ln(3+x)=\int\frac{1}{3+x}\,\mathrm{d}x=\int\sum_{n=0}^{\infty}(-1)^n\frac{1}{3^{n+1}}x^n\,\mathrm{d}x=C+\sum_{n=0}^{\infty}(-1)^n\frac{1}{3^{n+1}}\frac{x^{n+1}}{n+1}.$$

We still need to compute C which can be done by letting $x=0$ to get

$$\ln 3=C+\sum_{n=0}^{\infty}0=C.$$

Lastly, since integration maintains the radius of convergence, we still have that $R=3$. Putting everything together, we have

$$\boxed{\ln(3+x)=\ln 3+\sum_{n=0}^{\infty}(-1)^n\frac{1}{3^{n+1}}\frac{x^{n+1}}{n+1}, \qquad R=3.}$$

(d) Using our answer from part (c) and noting that the radius of convergence will be unchanged, we have

$$x^2\ln(3+x)=x^2\left(\ln 3+\sum_{n=0}^{\infty}(-1)^n\frac{1}{3^{n+1}}\frac{x^{n+1}}{n+1}\right)=\boxed{x^2\ln 3+\sum_{n=0}^{\infty}(-1)^n\frac{1}{3^{n+1}}\frac{x^{n+3}}{n+1}, \qquad R=3.}$$