Answer the following problems and simplify your answers.

1. (18pts) Find the **explicit solution** to the following initial value problem:

$$\begin{cases} \dfrac{\mathrm{d}z}{\mathrm{d}t} - e^{t+z} = 0 \\ z(0) = \ln 2 \end{cases}$$

Solution: Using separation of variables, we have

$$\int e^{-z}\,\mathrm{d}z = \int e^{t}\,\mathrm{d}t \implies -e^{-z} = e^{t} + C^*.$$

Solving for z yields

$$z = -\ln(C - e^{t}), \qquad C = -C^*.$$

Applying initial conditions, we have

$$\ln 2 = -\ln(C-1) \implies C = \frac{3}{2}.$$

Then, putting everything together, we have

$$\boxed{z = -\ln\left(\frac{3}{2} - e^{t}\right).}$$

2. (18 pts) Conisder the curve $y = \dfrac{x^3}{6} + \dfrac{1}{2x}$ on the interval $\dfrac{1}{2} \leq x \leq 1$.

   (a) Find the area of the surface obtained by rotating the curve about the y-axis.

   (b) Set up, **but do not evaluate**, the integral with respect to x to find the area of the surface rotated about $y = -2$.

Solution:

(a) First, we compute

$$y' = \frac{x^2}{2} - \frac{1}{2x^2}.$$

Next, we can compute our length element as

$$\begin{aligned} \mathrm{d}s = \sqrt{1 + (y')^2}\,\mathrm{d}x &= \sqrt{1 + \left(\frac{x^2}{2} - \frac{1}{2x^2}\right)^2}\,\mathrm{d}x \\ &= \sqrt{1 + \left(\frac{x^2}{2}\right)^2 - \frac{1}{2} + \left(\frac{1}{2x^2}\right)^2}\,\mathrm{d}x \\ &= \sqrt{\left(\frac{x^2}{2}\right)^2 + \frac{1}{2} + \left(\frac{1}{2x^2}\right)^2}\,\mathrm{d}x \\ &= \sqrt{\left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2}\,\mathrm{d}x \\ &= \frac{x^2}{2} + \frac{1}{2x^2}\,\mathrm{d}x. \end{aligned}$$

Since we are rotating the curve about the y-axis, the radius is $r = x$. Then the area is given by

$$\begin{aligned} A = \int_{1/2}^{1} 2\pi r\,\mathrm{d}s &= \int_{1/2}^{1} 2\pi x\left(\frac{x^2}{2} + \frac{1}{2x^2}\right)\mathrm{d}x \\ &= \pi\int_{1/2}^{1} x^3 + \frac{1}{x}\,\mathrm{d}x \\ &= \pi\left(\frac{1}{4}x^4 + \ln|x|\right)\Big|_{1/2}^{1} \\ &= \pi\left(\frac{1}{4} + \ln(1) - \frac{1}{64} - \ln(1/2)\right) \\ &= \boxed{\pi\left(\frac{15}{64} + \ln 2\right).} \end{aligned}$$

(b) The setup will be the same as in part (a) but this time, the radius will be

$$r = y - (-2) = \frac{x^3}{6} + \frac{1}{2x} + 2.$$

Then, the area integral is setup as

$$A = \int_{1/2}^{1} 2\pi r\,\mathrm{d}s = \boxed{\int_{1/2}^{1} 2\pi\left(\frac{x^3}{6} + \frac{1}{2x} + 2\right)\left(\frac{x^2}{2} + \frac{1}{2x^2}\right)\mathrm{d}x.}$$

3. (40 pts) Consider the region $\mathcal{R}$ bounded by $y = \frac{1}{2}x^2$ and $y = \sqrt{2x}$.

   (a) Sketch and shade $\mathcal{R}$, labeling the axes, intersections points, and curves.

   (b) Set up, **but do not evaluate**, integrals to find the following quanities **with respect to dx**:

      i. The volume of the solid generated by rotating $\mathcal{R}$ about $x = -1$.

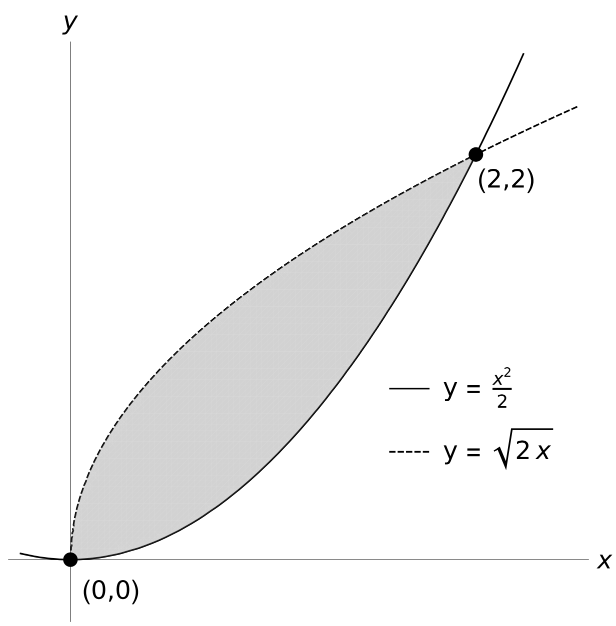
      ii. The volume of the solid generated by rotating $\mathcal{R}$ about $y = 2$.

      iii. The volume with a base of $\mathcal{R}$ and rectangular cross-sections perpendicular to the x-axis that have a height 3 times the length of their base.

   (c) Assuming a uniform density ρ, find the y-coordinate of the centroid of $\mathcal{R}$. **Fully simplify your answer.**

Solution:

(a) Graphing the region yields



(b) i. To rotate $\mathcal{R}$ about $x = -1$ using dx, we need cylindrical shells. In this case, the radius and height are given by

$$r = x - (-1) = x + 1, \qquad h = \sqrt{2x} - \frac{x^2}{2}$$

respectively. Using these values, the volume can be computed as

$$\boxed{V = \int_0^2 2\pi r h \, \mathrm{d}x = \int_0^2 2\pi(x+1)\left(\sqrt{2x} - \frac{x^2}{2}\right) \mathrm{d}x.}$$

ii. To rotate $\mathcal{R}$ about $y = 2$ using dx, we need to use the washer method. The outer radius and inner radius are given by

$$R = 2 - \frac{x^2}{2}, \qquad r = 2 - \sqrt{2x}$$

respectively. Plugging these values in the washer method formula gives our volume as

$$\boxed{V = \int_0^2 \pi(R^2 - r^2)\,\mathrm{d}x = \int_0^2 \pi\left(\left(2 - \frac{x^2}{2}\right)^2 - (2 - \sqrt{2x})^2\right)\mathrm{d}x.}$$

iii. The base length of each rectangle is given by the vertical distance in $\mathcal{R}$. In this case, the base

$$b = \sqrt{2x} - \frac{x^2}{2}.$$

Then, the height of the region is $h = 3b$ meaning the area of each rectangle is

$$A(x) = b \cdot h = 3\left(\sqrt{2x} - \frac{x^2}{2}\right)^2.$$

With this area function, our volume can be computed as

$$\boxed{V = \int_0^2 A(x)\,\mathrm{d}x = \int_0^2 3\left(\sqrt{2x} - \frac{x^2}{2}\right)^2 \mathrm{d}x.}$$

(c) To compute the y-coordinate of the centroid, we need the total mass m of $\mathcal{R}$ and the moment of $\mathcal{R}$ about x-axis. Computing our quantities, we have

$$m = \rho\int_0^2 \sqrt{2x} - \frac{x^2}{2}\,\mathrm{d}x = \rho\left(\frac{2\sqrt{2}}{3}x^{3/2} - \frac{x^3}{6}\right)\Bigg|_0^2 = \rho\left(\frac{2\sqrt{2}}{3}2^{3/2} - \frac{2^3}{6}\right) = \rho\frac{4}{3}$$

and

$$M_x = \rho\int_0^2 \frac{1}{2}\left((\sqrt{2x})^2 - \left(\frac{x^2}{2}\right)^2\right)\mathrm{d}x = \rho\int_0^2 \frac{1}{2}\left(2x - \frac{x^4}{4}\right)\mathrm{d}x = \rho\frac{1}{2}\left(x^2 - \frac{x^5}{20}\right)\Bigg|_0^2 = \rho\frac{6}{5}.$$

Then, the y-coordinate is given by

$$\boxed{\overline{y} = \frac{M_x}{m} = \frac{\rho \cdot 6/5}{\rho \cdot 4/3} = \frac{9}{10}.}$$

Not that it's needed but by symmetry of $\mathcal{R}$, we will also have $\overline{x} = \frac{9}{10}$.

4. (24 pts) Determine whether or not the following sequences converge or diverge. Justify your answer! If the sequence converges, find its limit.

(a) $\left\{\dfrac{(-1)^{n+1}n}{n^{3/2}+\sqrt{n}}\right\}$ (b) $\left\{\ln(2n^2+1)-2\ln(n+1)\right\}$ (c) $\left\{1+4^n\cdot 3^{2-n}\right\}$

Solution:

(a) First, we compute

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}n}{n^{3/2}+\sqrt{n}}\right| = \lim_{n\to\infty}\frac{n}{n^{3/2}+\sqrt{n}} = \lim_{n\to\infty}\frac{1/\sqrt{n}}{1+1/n} = \frac{0}{1+0} = 0$$

Since the absolute value of the sequence converges to zero,

$$\boxed{\lim_{n\to\infty}\frac{(-1)^{n+1}n}{n^{3/2}+\sqrt{n}} = 0.}$$

Finally, since the limit exists and is finite, the sequence $\boxed{\textit{converges.}}$

(b) Using log rules and continuity, we can compute our limit as

$$\lim_{n\to\infty}(\ln(2n^2+1)-2\ln(n+1)) = \lim_{n\to\infty}\ln\frac{2n^2+1}{(n+1)^2} = \ln\lim_{n\to\infty}\frac{2+1/n^2}{(1+1/n)^2} = \boxed{\ln 2.}$$

Since the limit exists and is finite, the sequence $\boxed{\textit{converges}}$.

(c) A little algebra yields

$$1+4^n\cdot 3^{2-n} = 1+3^2\frac{4^n}{3^n} = 1+9\left(\frac{4}{3}\right)^n.$$

The last term in our sequence is geometric with $r=4/3$. Since $4/3>1$,

$$\left(\frac{4}{3}\right)^n \to \infty \qquad \text{as } n\to\infty$$

meaning the original sequence $\boxed{\textit{diverges}}$ to infinity.