**Answer the following problems and simplify your answers.**

1. (36 pts) Evaluate the following integrals. **Show all work!**

(a) $\int x^3 \ln x \, dx$ (b) $\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}} \, dx$ (c) $\int \frac{2x}{x^2-4x+3} \, dx$

**Solution:**

(a) The integral follows via IBP with

$$u = \ln x \qquad v = \frac{1}{4}x^4 \qquad du = \frac{1}{x}\, dx \qquad dv = x^3 \, dx.$$

Then

$$\begin{aligned} \int x^3 \ln x \, dx &\overset{\text{IBP}}{=} \frac{1}{4}x^4 \ln x - \frac{1}{4}\int x^4 \frac{1}{x}\, dx \\ &= \boxed{\frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C.} \end{aligned}$$

(b) To start, we will use the trig substitution $x = 2\sin(\theta)$ which gives $dx = \cos\theta \, d\theta$ and $\sqrt{4-x^2} = 2\cos\theta$. Plugging this substitution in and changing bounds gives

$$\begin{aligned} \int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}}\, dx &= \int_0^{\pi/4} \frac{4\sin^2\theta}{2\cos\theta} 2\cos\theta \, d\theta \\ &= \int_0^{\pi/4} 4\sin^2\theta \, d\theta \\ &= \int_0^{\pi/4} 2(1-\cos 2\theta)\, d\theta \\ &= 2\left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_0^{\pi/4} \\ &= \boxed{\frac{\pi}{2} - 1.} \end{aligned}$$

(c) We will start by using partial fractions

$$\frac{2x}{x^2-4x+3} = \frac{2x}{(x-1)(x-3)} = \frac{A}{x-1} + \frac{B}{x-3} = \frac{A(x-3)+B(x-1)}{(x-1)(x-3)}$$

which, after canceling denominators, implies

$$2x = A(x-3) + B(x-1)$$

giving us the system of equations

$$\begin{aligned} A + B &= 2 \\ -3A - B &= 0. \end{aligned}$$

Solving this system yields $A = -1$ and $B = 3$. Plugging back in,

$$\begin{aligned} \int \frac{2x}{x^2-4x+3}\, dx &= \int -\frac{1}{x-1} + \frac{3}{x-3}\, dx \\ &= \boxed{-\ln|x-1| + 3\ln|x-3| + C.} \end{aligned}$$

2. (24 pts) With $I = \int_{-\pi/2}^{\pi/2} \cos x \, dx$, answer the following. **Leave your answer in exact form.**
   (a) Compute $T_4$ to estimate $I$.
   (b) Find a reasonable bound for the error $|E_T|$ of your calculation in part (a).
   (c) What is the smallest value of $n$ that will guarantee the error $|E_T|$ is less than $10^{-5}$?

**Solution:**

(a) To start, we have that $a = -\pi/2, b = \pi/2, n = 4, f(x) = \cos(x)$, and $\Delta x = \pi/4$. Computing the necessary $x$-values and function values gives

| $i$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $x_i$ | $-\pi/2$ | $-\pi/4$ | 0 | $\pi/4$ | $\pi/2$ |
| $f(x_i)$ | 0 | $\frac{\sqrt{2}}{2}$ | 1 | $\frac{\sqrt{2}}{2}$ | 0 |

Using our table, and the formula for the Trapezoidal rule, we have

$$T_4 = \frac{\pi}{8}\left(0 + 2\frac{\sqrt{2}}{2} + 2 \cdot 1 + 2\frac{\sqrt{2}}{2} + 0\right) = \boxed{\frac{\pi}{4}\left(\sqrt{2}+1\right).}$$

(b) To compute a reasonable error bound, we need to compute

$$f'(x) = -\sin(x) \implies \left|f''(x)\right| = |\cos(x)| \leq 1 = K.$$

Using the Trapezoidal error bound, we have

$$|E_T| \leq \frac{1(\pi)^3}{12 \cdot 4^2} = \boxed{\frac{\pi^3}{192}.}$$

(c) Using the same process as in part (b), we have

$$|E_T| \leq \frac{\pi^3}{12n^2} \leq 10^{-5}$$

which implies

$$\boxed{n \geq \sqrt{\frac{\pi^3}{12 \cdot 10^{-5}}}.}$$

3. (20 pts) Do the following integrals converge or diverge? Evaluate the convergent integrals.

   (a) $\displaystyle\int_0^\infty \frac{2e^{2x}}{1+e^{4x}}\,\mathrm{d}x$

   (b) $\displaystyle\int_2^\infty \frac{1+\cos^2 x}{x-1}\,\mathrm{d}x$

**Solution:**

(a) By direct calculation, we have

$$\begin{aligned}\int_0^\infty \frac{2e^{2x}}{1+e^{4x}}\,\mathrm{d}x &= \lim_{t\to\infty}\int_0^t \frac{2e^2x}{1+(e^{2x})^2}\,\mathrm{d}x \qquad\qquad u=e^{2x}, \mathrm{d}u = 2e^{2x}\,\mathrm{d}x\\ &= \lim_{t\to\infty}\int_0^{e^{2t}} \frac{1}{1+u^2}\,\mathrm{d}u\\ &= \lim_{t\to\infty} \arctan(u)\Big|_1^{e^{2t}}\\ &= \lim_{t\to\infty} \arctan(e^{2t}) - \frac{\pi}{4}\\ &= \boxed{\frac{\pi}{4}.}\end{aligned}$$

Since the limit is finite, the integral converges.

(b) We suspect that this integral diverges (since we are dividing by $x$) so let's try to show that via the comparison test. We can find a useful inequality as

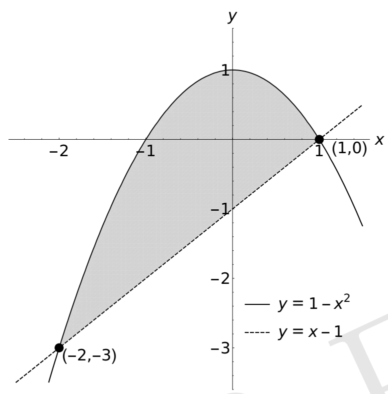
$$\frac{1+\cos^2 x}{x-1} \geq \frac{1}{x-1} \geq \frac{1}{x}.$$

Since $\int_2^\infty \frac{1}{x}\,\mathrm{d}x$ is a divergent p-integral (i.e. $p = 1 \leq 1$), the Comparison test tells us that $\int_2^\infty \frac{1+\cos^2 x}{x-1}\,\mathrm{d}x$ is also divergent.

4. (20 pts) Let $\mathcal{R}$ be the region bounded by $y = 1 - x^2$ and $y = x - 1$.
   (a) Sketch and shade the region $\mathcal{R}$. Label all axes, curves, and intersection points.
   (b) Set up, **but do not evaluate**, integrals to determine each of the following:
      i. The area of $\mathcal{R}$ using integration with respect to $x$.
      ii. The area of $\mathcal{R}$ using integration with respect to $y$.

**Solution:**

(a) Graphing our equations, intersection points, and shading $\mathcal{R}$, gives us




(b) Using the graph, our integrals are as follows:
   i. From the graph, our integral is

$$\boxed{A = \int_{-2}^{1} (1 - x^2) - (x - 1)\,\mathrm{d}x.}$$

   ii. Rewriting our equations in terms of $y$ gives us $x = \pm\sqrt{1-y}$ and $x = y + 1$. Then, using our graph, we can compute the area as

$$\boxed{A = \int_{-3}^{0} (y + 1) - (-\sqrt{1-y})\,\mathrm{d}y + \int_{0}^{1} \sqrt{1-y} - (-\sqrt{1-y})\,\mathrm{d}y.}$$

**Trigonometric Identities**

$$\cos^2(x) = \frac{1}{2}(1 + \cos 2x) \quad \sin^2 x = \frac{1}{2}(1 - \cos 2x) \quad \sin 2x = 2 \sin x \cos x \quad \cos 2x = \cos^2 x - \sin^2 x$$

**Inverse Trigonometric Integral Identities**

$$\int \frac{\mathrm{d}u}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C, u^2 < a^2$$
$$\int \frac{\mathrm{d}u}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$
$$\int \frac{\mathrm{d}u}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{u}{a}\right) + C, u^2 > a^2$$

**Midpoint Rule**

$$\int_a^b f(x)\,\mathrm{d}x \approx \Delta x[f(\overline{x}_1) + f(\overline{x}_2) + \cdots + f(\overline{x}_n)],\ \Delta x = \frac{b-a}{n},\ \overline{x}_i = \frac{x_{i-1} + x_i}{2},\ |E_M| \leq \frac{K(b-a)^3}{24n^2}$$

**Trapezoidal Rule**

$$\int_a^b f(x)\,\mathrm{d}x \approx \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)],\ \Delta x = \frac{b-a}{n},\ |E_T| \leq \frac{K(b-a)^3}{12n^2}$$