1 Evaluate the following integrals. Be sure to simplify your answers.

(a) (14 points) $\int_{-2}^{2} \frac{1}{x^2} dx$

(b) (14 points) $\int t^5 \sin(t^3) dt$

Solution: (a) Since there is a discontinuity in the integrand at $x = 0$, this is an improper integral that we have to split into two integrals, i.e.,

$$\begin{aligned}\int_{-2}^{2} \frac{1}{x^2}\,\mathrm{d}x &= \lim_{s\to 0^-} \int_{-2}^{s} \frac{1}{x^2}\,\mathrm{d}x + \lim_{t\to 0^+} \int_{t}^{2} \frac{1}{x^2}\,\mathrm{d}x \\ &= \lim_{s\to 0^-} \left[-\frac{1}{x}\right]_{-2}^{s} + \lim_{t\to 0^+} \left[-\frac{1}{x}\right]_{t}^{2} \\ &= \lim_{s\to 0^-} \left(-\frac{1}{s} - \frac{1}{2}\right) + \lim_{t\to 0^+} \left(-\frac{1}{2} + \frac{1}{t}\right) \\ &= \lim_{s\to 0^-} -\frac{1}{s} - 1 + \lim_{t\to 0^+} \frac{1}{t}.\end{aligned}$$

Since both limits diverge to positive infinity, the improper integral also diverges to positive infinity.

(b) We perform integration by parts for this problem. In order to do so, we first perform a u-substitution, taking $u = t^3$ and $\mathrm{d}u = 3t^2\,\mathrm{d}t$ (this can be rearranged to be obtain $\mathrm{d}t = \frac{\mathrm{d}u}{3t^2}$). The integral then becomes

$$\begin{aligned}\int t^5 \sin\left(t^3\right)\,\mathrm{d}t &= \frac{1}{3}\int u \sin(u)\,\frac{t^2}{t^2}\,\mathrm{d}u \\ &= \frac{1}{3}\int u\sin(u)\,\mathrm{d}u\end{aligned}$$

We take $v = u$ and $\mathrm{d}v = \mathrm{d}u$, and take $\mathrm{d}w = \sin(u)\,\mathrm{d}u$ and $w = -\cos(u)$. Applying the integration by parts formula, we have that

$$\begin{aligned}\frac{1}{3}\int u\sin(u)\,\mathrm{d}u &= \frac{1}{3}\left(vw - \int w\,\mathrm{d}v\right) \\ &= \frac{1}{3}\left(-u\cos(u) + \int \cos(u)\,\mathrm{d}u\right) \\ &= \frac{1}{3}\left(-u\cos(u) + \sin(u)\right).\end{aligned}$$

Undoing the u-substitution from earlier, we obtain that

$$\int t^5 \sin\left(t^3\right)\,\mathrm{d}t = \frac{1}{3}\left(-t^3\cos\left(t^3\right) + \sin\left(t^3\right)\right).$$

2 Given the following differential equation $\frac{dy}{dx} = \frac{x^2}{y}$ with the initial conditions $y(0) = -5$.

(a) (8 points) Solve explicitly for for the general solution $y(x)$.

(b) (8 points) From the general solution $y(x)$ derived in part (a), solve for the constant of integration c.

Solution: (a) This is a separable ODE, which we can rearrange the ODE as follows:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2}{y} \implies y\,\mathrm{d}y = x^2\,\mathrm{d}x.$$

We now integrate both sides of the rearranged ODE:

$$\begin{aligned} \int y\,\mathrm{d}y = \int x^2\,\mathrm{d}x &\implies \frac{y^2}{2} + c = \frac{x^3}{3} \\ &\implies y^2 = \frac{2}{3}x^3 + c \\ &\implies y\,(x) = \pm\sqrt{\frac{2}{3}x^3 + c}. \end{aligned}$$

Since we take the square root of both sides, we have to consider both the negative and positive square root of the right-hand side.

(b) To determine the constant of integration c in the general solution from earlier, we impose the given initial condition. Since $y\,(0) = -5$, this implies that we take the negative square root of the right-hand side from earlier, i.e., we take $y\,(x) = -\sqrt{\frac{2}{3}x^3 + C}$. Imposing the initial condition on y, we have that

$$-5 = -\sqrt{\frac{2}{3}\,(0)^3 + C} \implies 25 = C.$$

Thus, the solution that satisfies the above ODE and the given initial condition is $y\,(x) = -\sqrt{\frac{2}{3}x^3 + 25}$.

3 (10 points) Consider the curve defined by $y = \sec(x)$ on $0 \le x \le \frac{\pi}{4}$. Set up but **do not evaluate** the surface area of the solid obtained by rotating the curve about the **y-axis**.

Solution: Since we are rotating the above curve about the y-axis, we use the following surface area of a solid of revolution formula:

$$SA = 2\pi \int_a^b r\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x.$$

Here, r is the distance from the axis of rotation to the curve, which is simply x in this case. The lower and upper bounds of integration are obtained by taking the lower and upper bounds on x that are provided, i.e.,

$$\begin{aligned} a &= 0, \\ b &= \frac{\pi}{4}, \end{aligned}$$

It remains to compute $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$, which we do as follows:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \sec(x)\tan(x),\\ \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 &= \sec^2(x)\tan^2(x).\end{aligned}$$

Substituting the above into the surface formula from earlier, we have that the surface area of the solid of revolution is given by

$$SA = 2\pi\int_0^{\frac{\pi}{4}} x\sqrt{1+\sec^2(x)\tan^2(x)}\,\mathrm{d}x.$$

4 Does the sequence or series converge? If so, what does it converge to? Justify your answer and name any tests or theorems you use.

Solution: (a) (12 points) $a_n = \dfrac{3^{n+2}}{5^n}$

We first note that

$$\begin{aligned}a_n &= \frac{3^{n+2}}{5^n}\\ &= \left(3^2\right)\frac{3^n}{5^n}\\ &= 9\left(\frac{3}{5}\right)^n.\end{aligned}$$

We then take the limit as n goes to infinity of the a_n's:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 9\left(\frac{3}{5}\right)^n.$$

Since $\left|\frac{3}{5}\right| < 1$, the above limit goes to 0, i.e,, the sequence $\{a_n\}$ converges to 0.

(b) (12 points) $\displaystyle\sum_{n=2}^{\infty}\frac{\sqrt{n}}{\ln(n)}$ $\Longrightarrow$ Applying L'Hospital's Rule $\Longrightarrow$

$\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{\sqrt{n}}{\ln(n)} = \lim_{n\to\infty}\frac{\frac{1}{2\sqrt{n}}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n}{2\sqrt{n}} = \lim_{n\to\infty}\frac{\sqrt{n}}{2} = \infty \neq 0 \therefore$ it diverges by the divergence test. Could also do a DCT and it diverges by the p-series $\dfrac{1}{\sqrt{n}}$.

5 Consider the function $f(x) = \dfrac{1}{1+x^2}$.

(a) (12 points) Write the function $f(x)$ as a Maclauren series.

(b) (12 points) Using your answer from part (a), write $g(x) = \tan^{-1}(x)$ as a Maclauren Series.

Solution: (a) $\displaystyle f(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n(x)^{2n}$

(b) $\int f(x)dx = \int \frac{1}{1+x^2}dx = \tan^{-1}(x) = g(x) \therefore \tan^{-1}(x) = \sum_{n=0}^{\infty} \int (-1)^n (x)^{2n} dx \therefore g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n+1}}{2n+1}$

6 (a) (12 points) Find the tangent line to the parametric curve $x = te^{t^2} + 1, y = \cos^2(t) - 2t$, at the point $t = 0$.

(b) (12 points) Consider the parametric cuve $x = \sin^2(t), y = \sin(3t)$ bounded by $0 \leq t \leq \frac{\pi}{3}$. Set-up but **do not evaluate** the area under the curve using a parametric integral.

Solution: (a) The point on the curve has Cartesian coordinate $(1, 1)$. The equation of the tangent line will posesses the form $y = mx + b$, where m is the slope of the line and b is the $y - intercept$. To find the slope, we calculate the derivative as $\left.\frac{dy}{dx}\right|_{(1,1)} = \left.\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right|_{t=0} = \left.\frac{-2\cos(t)\sin(t) - 2}{e^{t^2} + 2t^2e^{t^2}}\right|_{t=0} = -2.$

So we have $y = 2x + b$. To solve for b, we use the fact that the tangent line passes through the curve $(1, 1)$, implying that $1 = -2 + b$, so $b = 3$. Thus, the equation for the tangent line is $y = -2x + 3$.

(b) $\int_0^{\frac{\pi}{3}} y dx = 2 \int_0^{\frac{\pi}{3}} \sin(3t)\sin(t)\cos(t)$

7 (a) (12 points) Consider the polar curve $r = \theta + \sin\theta$ for $0 \leq \theta \leq \frac{\pi}{2}$. Set-up but **do not evaluate** an integral to find the length of the polar curve.

(b) (12 points) Set-up but **do not evaluate** the area of one rose petal, given by the equation $r = \cos(3\theta)$.

Solution: (a) $L = \int_0^{\frac{\pi}{2}} \sqrt{(\theta + \sin(\theta))^2 + (1 + \cos(\theta))^2} d\theta$

(b) One petal is drawn for $\theta \in [\frac{\pi}{6}, \frac{\pi}{6}]$. Use symmetry to get the integral $2 \int_0^{\frac{\pi}{6}} \frac{1}{2} \cos^2(3\theta) d\theta$