1. Determine if the series converge or diverge. Be sure to fully justify your answer and state what test that you used.

   (a) (8 points) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{3n-1}$

   (b) (8 points) $\displaystyle\sum_{n=1}^{\infty} \frac{5}{6^{n-1}}$

   (c) (8 points) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{5n-2n^3}{6n^3+3}\right)^n$

Solution: (a) We apply the divergence test to this series:

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{n}{3n-1} \\ &= \lim_{n\to\infty} \frac{n}{3n-1} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} \\ &= \lim_{n\to\infty} \frac{1}{3-\frac{1}{n}} \\ &= \frac{1}{3} \\ &\neq 0.\end{aligned}$$

Since the terms of the series converge to $\frac{1}{3}$, the series diverges by the divergence test.

(b) We first note that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{5}{6^{n-1}} &= 5\sum_{n=1}^{\infty} \frac{1}{6^{-1}} \cdot \frac{1}{6^n} \\ &= 30\sum_{n=1}^{\infty} \left(\frac{1}{6}\right)^n.\end{aligned}$$

This is a geometric series whose common ratio is $r = \frac{1}{6}$. Since $|r| = \left|\frac{1}{6}\right| < 1$, this series converges by the geometric series test.

Equivalently, we can apply the ratio test or root test to the series and show that the limit is less than 1 in magnitude.

(c) We apply the root test to this series:

$$\begin{aligned}
\lim_{n\to\infty} \sqrt[n]{|a_n|} &= \lim_{n\to\infty} \left| \left(\frac{5n-2n^3}{6n^3+3} \right)^n \right|^{\frac{1}{n}} \\
&= \lim_{n\to\infty} \left| \left(\frac{5n-2n^3}{6n^3+3} \right) \right| \\
&= \lim_{n\to\infty} \left| \left(\frac{5n-2n^3}{6n^3+3} \right) \cdot \frac{\frac{1}{n^3}}{\frac{1}{n^3}} \right| \\
&= \lim_{n\to\infty} \left| \left(\frac{\frac{5}{n^2}-2}{6+\frac{3}{n^3}} \right) \right| \\
&= \lim_{n\to\infty} \left| -\frac{2}{6} \right| \\
&= \frac{1}{3} \\
&< 1.
\end{aligned}$$

Since $\frac{1}{3} < 1$, the series absolutely converges by the root test.

2. Determine the interval of convergence and the radius of convergence for the following power series.

(a) (15 points) $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$

(b) (15 points) $\displaystyle\sum_{n=1}^{\infty} \frac{(x+2)^n}{n!}$

Solution: (a) By inspection, we see that the center of this power series is $a = 0$. We can apply the ratio test to this series to determine its radius of convergence and interval of convergence:

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(-3)^{n+1}x^{n+1}}{\sqrt{(n+1)+1}} \cdot \frac{\sqrt{n+1}}{(-3)^n x^n} \right| \\
&= \lim_{n\to\infty} 3\,|x| \left| \frac{\sqrt{n+1}}{\sqrt{n+2}} \right| \\
&= \lim_{n\to\infty} 3\,|x| \left| \frac{\sqrt{n+1}}{\sqrt{n+2}} \cdot \frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} \right| \\
&= \lim_{n\to\infty} 3\,|x| \left| \frac{\sqrt{1+\frac{1}{\sqrt{n}}}}{\sqrt{1+\frac{2}{\sqrt{n}}}} \right| \\
&= 3\,|x|\,.
\end{aligned}$$

For this series to absolutely converge, we require that

$$\begin{aligned} 3|x| < 1 &\implies -1 < 3x < 1 \\ &\implies -\frac{1}{3} < x < \frac{1}{3}. \end{aligned}$$

From this, we see that the radius of convergence is $R = \frac{1}{3}$ and that the tentative interval of convergence is $I = \left(-\frac{1}{3}, \frac{1}{3}\right)$. We need to also check the behavior of the series at the endpoints of the above tentative interval of convergence:

i. $x = -\frac{1}{3}$

At this endpoint, the series evaluates to

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(-3)^n \left(-\frac{1}{3}\right)^n}{\sqrt{n+1}} &= \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}} \\ &= \sum_{n=1}^{\infty} \frac{1}{(n+1)^{\frac{1}{2}}} \end{aligned}$$

To determine the convergence of this series, we apply the limit comparison test with the p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}},$$

which we know is a divergent p-series since $p = \frac{1}{2} < 1$:

$$\begin{aligned} \lim_{n\to\infty} \frac{\frac{1}{(n+1)^{\frac{1}{2}}}}{\frac{1}{n^{\frac{1}{2}}}} &= \lim_{n\to\infty} \frac{n^{\frac{1}{2}}}{(n+1)^{\frac{1}{2}}} \\ &= \lim_{n\to\infty} \frac{n^{\frac{1}{2}}}{(n+1)^{\frac{1}{2}}} \cdot \frac{\frac{1}{n^{\frac{1}{2}}}}{\frac{1}{n^{\frac{1}{2}}}} \\ &= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^{\frac{1}{2}}} \\ &= 1 \\ &> 0. \end{aligned}$$

Since this limit is greater than 0 and finite, the original series diverges at $x = -\frac{1}{3}$ by the limit comparison test.

ii. $x = \frac{1}{3}$

At this endpoint, the series evaluates to

$$\sum_{n=1}^{\infty} \frac{(-3)^n \left(\frac{1}{3}\right)^n}{\sqrt{n+1}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$$
$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)^{\frac{1}{2}}}$$

This is an alternating series, where the positive portion of the terms are given by $b_n = \frac{1}{(n+1)^{\frac{1}{2}}}$. We apply the alternating series test to determine the convergence of the series:

A. $b_n \geq 0$ for all $n \geq 1$

   By inspection, we see that $b_n \geq 0$ for all $n \geq 1$.

B. $\{b_n\}_{n=1}^{\infty}$ is a decreasing sequence

   By looking at the continuous function, $f(x) = \frac{1}{\sqrt{n+1}}$, we see that $f'(x) = -\frac{1}{2}(n+1)^{-\frac{3}{2}}$, which implies that f is a decreasing function. Thus, the b_n's are also decreasing.

C. $\lim_{n\to\infty} b_n = 0$

   We see that

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}}$$
$$= 0.$$

Thus, all the conditions of the alternating series test are satisfied, so this is a convergent alternating series.

Thus, the interval of convergence for this series is $\left(-\frac{1}{3}, \frac{1}{3}\right)$.

(b) By inspection, we see that the center of this power series is $a = -2$. We can apply the ratio test to this series to determine its radius of convergence and interval of convergence:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x+2)^{n+1}}{(n+1)!} \cdot \frac{n!}{(x+2)^n}\right|$$
$$= \lim_{n\to\infty} |x+2| \left|\frac{1}{n+1}\right|$$
$$= |x+2| \lim_{n\to\infty} \left|\frac{1}{n+1}\right|$$
$$= 0.$$

From this, we see that the radius of convergence is $R = \infty$ and the interval of convergence is $I = (-\infty, \infty)$.

3. (a) 10 points) Start with the Maclauren Series for $\frac{1}{1-x}$ to find a power series representation for $\frac{1}{1+2x^2}$. Show all work.
(b) (8 points) Use your answer from part (a) to find its interval of convergence.

Solution:

(a) $\frac{1}{1+2x^2} = \frac{1}{1-(-2x^2)} \therefore \Longrightarrow$ from the Maclauren Series that

$$\frac{1}{1+2x^2} = \sum_{n=0}^{\infty}(-2x^2)^n = \sum_{n=0}^{\infty}(-1)^n 2^n x^{2n}$$

(b) Since the equality $\frac{1}{1-r} = \sum r^n$ is only valid for $|r| < 1$, we see that the series converges for $|-2x^2| < 1$, meaning that $|x| < \frac{1}{\sqrt{2}}$ (we could have also seen this using the ratio test). Therefore the interval of convergence is $(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. Testing each endpoint ultimately shows that it also diverges at both end points.

4. (12 points) Find the first 4 terms $(c_0 + c_1x + c_2x^2 + c_3x^3)$ of the Maclauren series for $\cos(x + \frac{\pi}{2})$

Solution: The goal here is to find the first 3 derivatives at:
$x = 0 \Longrightarrow$ -sin(x + $\frac{\pi}{2}$), $-\cos(x + \frac{\pi}{2}), \sin(\pi + \frac{\pi}{2})$. The values of the functions at the derivatives at $0, -1, 0, 1$ and we get $\Longrightarrow \boxed{0 - x + 0 + \frac{1}{3!}x^3}$

5. (a) (10 points) Evaluate $\int_0^{0.4} \ln(1+x)dx$ as an infinite series.
(b) (6 points) Estimate the error if you use the first 2 terms of the series in 5 (a) to approximate the value of the definite integral.

Solution: (a) $\int_0^{0.4} \ln(1+x)dx = \int_0^{0.4} \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}dx = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{n+1}}{n(n+1)}\Bigg|_0^{0.4}$

$$= \boxed{\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(0.4)^{n+1}}{n^2+n}}$$

(b) The First term at $n = 1$ yields $\frac{(0.4)^2}{2}$. The second term at $n = 2$ yields $\frac{(-0.4)^3}{6}$. Adding these two terms together yields, $\boxed{\frac{(0.4)^2}{2} + \frac{(-0.4)^3}{6} = \frac{26}{375}}$ and at $n = 3 = \frac{(0.4)^4}{12}$

$$\therefore |R_n| = |S - Sn| \leq b_{n+1} \Longrightarrow |S - S_2| \leq b_3 \Longrightarrow \boxed{|Rn| = |S - \frac{26}{375}| \leq \frac{(0.4)^4}{12}}$$