1. Let $\mathcal{R}$ be the region bounded by the curve $y = x^2$ and $y = x$. The region $\mathcal{R}$ is rotated around the line $x = 1$ to form a solid.

   (a) (12 points) Set up an integral(s) for the volume of this solid using the Method of Cylindrical Shells. **EVALUATE THE INTEGRAL.**

   SOLUTION:

$$\boxed{\int_0^1 2\pi(x - x^2)(1 - x)dx = \frac{\pi}{6}}$$

   (b) (8 points) Set up an integral(s) for the volume of this solid using the Disk/Washer Method. **DO NOT EVALUATE THE INTEGRAL.**

   SOLUTION:

$$V_{washer} = \boxed{\int_0^1 \pi[(1 - y^2)^2 - (1 - \sqrt{y})^2]dy}$$

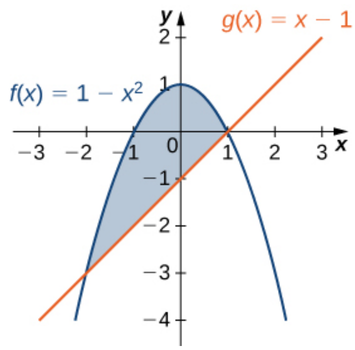
2. (15 points) Consider the region bounded by $y = e^{x^2}$, $x = 0$, $y = 0$, and $x = 3$. Set up but **DO NOT EVALUATE** the surface area of the solid obtained by rotating the region about the **x-axis**.

SOLUTION:

$$S = \int_a^b 2\pi f(x)\sqrt{1 + f'(x)^2}\, dx$$

$$\begin{aligned}\text{Surface Area} &= 2\pi \int_0^2 e^{x^2} \sqrt{1 + \left[\frac{d}{dx}(e^{x2})\right]^2} dx \\ &= \boxed{2\pi \int_0^2 e^{x^2}\sqrt{1 + 4x^2 e^{2x^2}} dx}\end{aligned}$$

3. (15 points) Consider the region of uniform density $\rho = 1$ bounded above by the function $f(x) = 1 - x^2$ and below by the the function $g(x) = x - 1$. Find **just the x-coordinate** for the centroid of the region.



SOLUTION:

The graphs of the function intersect at (-2,-3) and (1,0), so we integrate from (-2,1). First, we need to calculate the total mass:

$$m = \rho \int_a^b [f(x) - g(x)]dx = \int_{-2}^{1} [1 - x^2 - (x-1)]dx = \int_{-2}^{1} (2 - x^2 - x)dx$$

$$= [2x - \frac{1}{3}x^3 - \frac{1}{2}x^2]|_{-2}^{1} = [2 - \frac{1}{3} - \frac{1}{2} - [-4 + \frac{8}{3} - 2] = \frac{9}{2}$$

Next we compute the moments:

$$M_y = \rho \int_a^b x([f(x)] - [g(x)]dx$$

$$= \int_{-2}^{1} \rho x[(1 - x^2) - (x - 1)]dx = \frac{-9}{4}\rho$$

$$\bar{x} = \frac{M_y}{m} = \frac{-9}{4}\frac{2}{9} = \boxed{\frac{-1}{2}}$$

4. For each of the following, determine whether the sequence converges or diverges. Explain your work in each case.

(a) (6 points) $a_n = \frac{(\ln(n))^2}{n}$

SOLUTION:

Consider the function $f(x) = \frac{[\ln(x)]^2}{x}$. Then using the L'Hopital's Rule, we obtain

$$\begin{aligned}\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{(\ln(x))^2}{x} \\ &= \lim_{x\to\infty} \frac{2\ln(x)\frac{1}{x}}{1} \\ &= \lim_{x\to\infty} \frac{2\ln(x)}{1}\end{aligned}$$

Using the L'Hopital's Rule again, we obtain:

$$= \boxed{\lim_{x\to\infty} \frac{2\frac{1}{x}}{1} = 0}$$

(b) (6 points) $c_n = \tan^{-1}(2n)$

SOLUTION:

$$= \lim_{x\to\infty} \tan^{-1}(2x) = \boxed{\frac{\pi}{2}}$$

. Therefore the sequence $c_n = \tan^{-1}(2n)$ converges.

(c) (6 points) $a_n = 5 - \dfrac{3}{n^2}$

SOLUTION:

$$\lim_{n\to\infty} (5 - \frac{3}{n^2}) = 5.$$

Therefore the sequence converges and its limit is 5.

(d) (6 points) $b_n = \dfrac{n!}{n^n}$

SOLUTION:
If $\frac{a_{n+1}}{a_n} < 1$ then $a_{n+1} < a_n$. Therefore,

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}}\frac{n^n}{n!} = \frac{(n+1)!}{n!}\frac{n^n}{(n+1)^{n+1}} = \frac{n+1}{n+1}\left(\frac{n}{n+1}\right)^n = \left(\frac{n}{n+1}\right)^n < 1$$

Therefore the sequence $b_n = \dfrac{n!}{n^n}$ converges. Can also be shown using the **squeeze theorem**.

5. (12 points) If the work required to stretch a spring $1ft$ beyond its natural length is $12ft{-}lb$, how much work is needed to stretch it $9in.$ beyond its natural length?

SOLUTION:
Hooke's Law tells us that the force to stretch a spring x units beyond its natural length is $f(x) = kx$, where k is a positive constant. The phrase ..."1ft beyond its natural length..." tells us x is changing from $0ft$ to $1ft$. We also know that $W = \int_a^b f(x)\text{dx}$. Substituting the given information,

$$12 = \int_0^1 kx\text{dx}$$

.

$$12 = k[\frac{x^2}{2}]|_0^1 \implies k = 24$$

So our force function is $f(x) = 24x$, and, for this particular spring, we have $W = \int_a^b 24x\text{dx}$. The problem asks "..how much work is needed to stretch $9in$ beyond its natural length.." Our integral uses feet as the length unit, so we must realize that $9in = \frac{3}{4}ft$. The work is given by

$$W = \int_0^{\frac{3}{4}} 24x\text{dx} = \left(\frac{24x^2}{2}\Bigg|_0^{\frac{3}{4}}\right) = \left(24 \times \frac{(\frac{3}{4})^2}{2}\right) - \left(24 \times \frac{0^2}{2}\right) = \left(24 \times \frac{9}{32}\right) = \boxed{\frac{27}{4}}$$

6. (14 points) Solve the differential equation. Leave your final answer in **implicit** form for y. (Do not solve explicitly for y).

$$y' + \frac{1+y^3}{xy^2(1+x^2)} = 0$$

$$\implies \frac{dy}{dx} = -\frac{1+y^3}{xy^2(1+x^2}y^2$$

$$\frac{y^2}{1+y^3} = \frac{-dx}{x(1+x^2)}$$

$$\int \frac{y^2 dy}{1+y^3} = -\int \frac{dx}{x(1+x^2)}$$

LHS: $u = 1+y^3, du = 3y^2 dy \implies \frac{1}{3}\int \frac{3y^2 dy}{1+y^3} = \frac{1}{3}\ln(|1+y^3|)$

RHS: $\int \frac{dx}{x(1+x^2} \implies \text{PFD} \therefore \frac{1}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{1+x^2}$

$$\frac{1}{x(1+x^2} = \frac{A(1+x^2)}{x(1+x^2)} + \frac{(Bx+c)x}{x(1+x^2)}$$

$$1 = A(1+x^2) + Bx^2 + Cx$$

$$1 = A + Ax^2 + Bx^2 + cx$$

Solving for our coefficients, we obtain $A = 1, B = -1, C = 0$

$$\therefore \int \frac{dx}{x(1+x^2)} = \frac{dx}{x} + \int \frac{-xdx}{1+x^2} = \ln(|x|) - \frac{1}{2}\ln|1+x^2| + c$$

$$\therefore \boxed{\frac{1}{3}\ln(|1+y^3|) = -\ln(|x|) + \frac{1}{2}\ln|1+x^2| + c}$$

or $$\boxed{\ln(|1+y^3|)^{\frac{1}{3}} + \ln(|x|) - \ln(|1+x^2|)^{\frac{1}{2}} = c}$$

— END —