1. Let $R$ be the region bounded by the curve $y = x^2$ and $y = x$. The region $R$ is rotated around the line $x = 1$ to form a solid.

(a) (12 points) Set up an integral(s) for the volume of this solid using the Method of Cylindrical Shells. EVALUATE THE INTEGRAL.

SOLUTION:

$$\int_0^1 2\pi (x - x^2)(1 - x) \, dx = \frac{\pi}{6}$$

(b) (8 points) Set up an integral(s) for the volume of this solid using the Disk/Washer Method. DO NOT EVALUATE THE INTEGRAL.

SOLUTION:

$$V_{\text{washer}} = \int_0^1 \pi [(1 - y^2)^2 - (1 - \sqrt{y})^2] \, dy$$

2. (15 points) Consider the region bounded by $y = e^{x^2}$, $x = 0$, $y = 0$, and $x = 3$. Set up but DO NOT EVALUATE the surface area of the solid obtained by rotating the region about the $x$-axis.

SOLUTION:

$$S = \int_a^b 2\pi f(x) \sqrt{1 + f'(x)^2} \, dx$$

Surface Area = $2\pi \int_0^3 e^{x^2} \sqrt{1 + \left[ \frac{d}{dx} (e^{x^2}) \right]^2} \, dx$

$$= 2\pi \int_0^3 e^{x^2} \sqrt{1 + 4x^2 e^{2x^2}} \, dx$$
3. (15 points) Consider the region of uniform density $\rho = 1$ bounded above by the function $f(x) = 1 - x^2$ and below by the function $g(x) = x - 1$. Find just the x-coordinate for the centroid of the region.

**SOLUTION:**
The graphs of the function intersect at (-2,-3) and (1,0), so we integrate from (-2,1).

First, we need to calculate the total mass:

$$m = \rho \int_{-2}^{1} [f(x) - g(x)] dx = \int_{-2}^{1} [1 - x^2 - (x - 1)] dx = \int_{-2}^{1} (2 - x^2 - x) dx$$

$$= [2x - \frac{1}{3} x^3 - \frac{1}{2} x^2]_{-2}^{1} = [2 - \frac{1}{3} - \frac{1}{2} - 2 - \frac{8}{3} - 2] = \frac{9}{2}$$

Next we compute the moments:

$$M_y = \rho \int_{-2}^{1} x([f(x)] - [g(x)]) dx$$

$$= \int_{-2}^{1} \rho x[(1 - x^2) - (x - 1)] dx = \frac{-9}{4} \rho$$

$$\bar{x} = \frac{M_y}{m} = \frac{-9}{4 \cdot \frac{9}{2}} = \frac{-1}{2}$$
4. For each of the following, determine whether the sequence converges or diverges. Explain your work in each case.

(a) (6 points) \( a_n = \frac{(\ln(n))^2}{n} \)

SOLUTION:

Consider the function \( f(x) = \frac{[\ln(x)]^2}{x} \). Then using the L’Hopital’s Rule, we obtain

\[
\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{(\ln(x))^2}{x} = \lim_{x \to \infty} \frac{2\ln(x)}{1} = 0.
\]

(b) (6 points) \( c_n = \tan^{-1}(2n) \)

SOLUTION:

\[
= \lim_{x \to \infty} \tan^{-1}(2x) = \frac{\pi}{2}.
\]

Therefore the sequence \( c_n = \tan^{-1}(2n) \) converges.

(c) (6 points) \( a_n = 5 - \frac{3}{n^2} \)

SOLUTION:

\[
\lim_{n \to \infty} \left(5 - \frac{3}{n^2}\right) = 5.
\]

Therefore the sequence converges and its limit is 5.
(d) (6 points) \( b_n = \frac{n!}{n^n} \)

**SOLUTION:**

If \( \frac{a_{n+1}}{a_n} < 1 \) then \( a_{n+1} < a_n \). Therefore,

\[
\frac{a_{n+1}}{a_n} = \frac{(n+1)! \ n^n}{(n+1)^{n+1} \ n!} = \frac{(n+1)!}{n!} \frac{n^n}{(n+1)^{n+1}} = \frac{n+1}{n+1} \left( \frac{n}{n+1} \right)^n = \left( \frac{n}{n+1} \right)^n < 1
\]

Therefore the sequence \( b_n = \frac{n!}{n^n} \) converges. Can also be shown using the squeeze theorem.

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5. (12 points) If the work required to stretch a spring 1 ft beyond its natural length is 12 ft-lb, how much work is needed to stretch it 9 in. beyond its natural length?

**SOLUTION:**

Hooke’s Law tells us that the force to stretch a spring \( x \) units beyond its natural length is \( f(x) = kx \), where \( k \) is a positive constant. The phrase “1 ft beyond its natural length...” tells us \( x \) is changing from 0 ft to 1 ft. We also know that \( W = \int_{a}^{b} f(x) \, dx \). Substituting the given information,

\[
12 = \int_{0}^{1} kx \, dx
\]

So our force function is \( f(x) = 24x \), and, for this particular spring, we have \( W = \int_{0}^{b} 24x \, dx \). The problem asks ”...how much work is needed to stretch 9 in beyond its natural length...” Our integral uses feet as the length unit, so we must realize that 9 in = \( \frac{3}{4} \) ft. The work is given by

\[
W = \int_{0}^{\frac{3}{4}} 24x \, dx = \left( \frac{24x^2}{2} \bigg|_{0}^{\frac{3}{4}} \right) = \left( 24 \times \frac{\left(\frac{3}{4}\right)^2}{2} \right) - \left( 24 \times \frac{0^2}{2} \right) = \left( 24 \times \frac{9}{32} \right) = \frac{27}{4}
\]
6. (14 points) Solve the differential equation. Leave your final answer in implicit form for \( y \). (Do not solve explicitly for \( y \)).

\[
y' + \frac{1 + y^3}{xy^2(1 + x^2)} = 0
\]

\[
\Rightarrow \frac{dy}{dx} = -\frac{1 + y^3}{xy^2(1 + x^2)}
\]

\[
y^2 \frac{1}{1 + y^3} = -dx
\]

\[
\int y^2 dy \frac{1}{1 + y^3} = - \int \frac{dx}{x(1 + x^2)}
\]

**LHS:** \( u = 1 + y^3, du = 3y^2 dy \Rightarrow \frac{1}{3} \int \frac{3y^2 dy}{1 + y^3} = \frac{1}{3} \ln(|1 + y^3|) \)

**RHS:** \[
\int \frac{dx}{x(1 + x^2)} \Rightarrow \text{PFD} \quad \frac{1}{x(1 + x^2)} = \frac{A}{x} + \frac{Bx + C}{1 + x^2}
\]

\[
\frac{1}{x(1 + x^2)} = \frac{A(1 + x^2)}{x(1 + x^2)} + \frac{(Bx + c)x}{x(1 + x^2)} = 1 = A(1 + x^2) + Bx^2 + Cx
\]

\[
1 = A + Ax^2 + Bx^2 + Cx
\]

Solving for our coefficients, we obtain \( A = 1, B = -1, C = 0 \)

\[
\therefore \int \frac{dx}{x(1 + x^2)} = \frac{dx}{x} + \int \frac{-x dx}{1 + x^2} = \ln(|x|) - \frac{1}{2} \ln|1 + x^2| + c
\]

\[
\therefore \frac{1}{3} \ln(|1 + y^3|) = -\ln(|x|) + \frac{1}{2} \ln|1 + x^2| + c
\]

or \[
\ln(|1 + y^3|) = \frac{1}{3} \ln|1 + y^3| + \ln(|x|) - \ln(|1 + x^2|)^{\frac{1}{2}} = c
\]