1. Let $\mathcal{R}$ be the region bounded by $y = 2 - x$ and $y = x^2$

   (a) (5 points) Sketch the region. Be sure to label all axes, curves, and intersection points.

   SOLUTION:

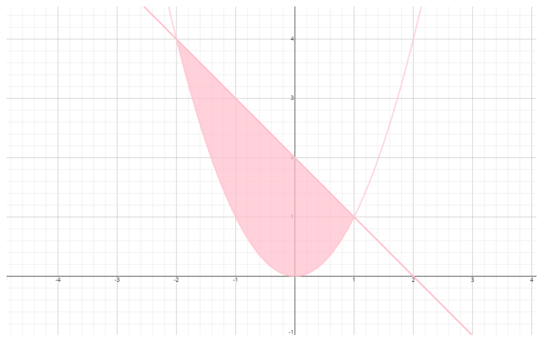


Figure 1: Bounded Area Region

   (b) (7 points) Set up the dx integral(s) (integral(s) with respect to x) which, if evaluated, would give the area of the region. **DO NOT EVALUATE**.

   SOLUTION:

$$\text{Area} = \int_{-2}^{1} (2 - x - x^2)dx$$

   (c) (7 points) Set up the dy integral(s) (integral(s) with respect to y) which, if evaluated, would give the area of the region. **DO NOT EVALUATE**.

   SOLUTION:

$$A = \int_0^1 2\sqrt{y}\,dy + \int_1^4 (2-y) - \sqrt{(y)}\,dy$$

   (d) (5 points) Find the area of the region $\mathcal{R}$ from either (b) or (c).

   SOLUTION:
   Evaluation of either (b) or (c) will yield a result of $\frac{9}{2}$

2. Consider the curve $y = \sin(x)$.

(a) (8 points) Use the Midpoint Rule with $n = 4$ sub-intervals to estimate the integral of this curve from $x = 0$ to $x = \pi$.

SOLUTION:

Here, we know that $n = 4, a = 0$, and $b = \pi$. Thus, we can say $\Delta x = \frac{(b-a)}{n} = \frac{\pi}{4}$. We now divide the interval [0,4] into $n = 4$ subintervals and of the length $\Delta x = \frac{\pi}{4}$ with the following endpoints: $a = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi = b$.
Now we just evaluate the function at the midpoints of the sub-intervals.
$f(\frac{x_0+x_1}{2}) = f(\frac{0+\frac{\pi}{4}}{2}) = f(\frac{\pi}{8})$
$f(\frac{x_1+x_2}{2}) = f(\frac{\frac{\pi}{4}+\frac{\pi}{2}}{2}) = f(\frac{3\pi}{8})$
$f(\frac{x_2+x_3}{2}) = f(\frac{\frac{\pi}{2}+\frac{3\pi}{4}}{2}) = f(\frac{5\pi}{8})$
$f(\frac{x_3+x_4}{2}) = f(\frac{\frac{3\pi}{4}+\pi}{2}) = f(\frac{7\pi}{8})$
Therefore $M_4 = \frac{\pi}{4}[\sin(\frac{\pi}{8}) + \sin(\frac{3\pi}{8}) + \sin(\frac{5\pi}{8}) + \sin(\frac{7\pi}{8})]$

(b) (8 points) Estimate the error in the calculation from part (a)

SOLUTION:

To estimate the error, we can do it with the bounding formula,

$$|E_M| \leq \frac{k(b-a)^3}{24n^2}$$

where k is the maximum value of $|\frac{d^2y}{dx^2}|$ on $x \in [0, \pi]$.
$\therefore |E_M| \leq \frac{k(b-a)^3}{24n^2}$
$= \frac{k\cdot\pi^3}{24\cdot4^2}$we can find k now,$\frac{d^2y}{dx^2} = -\sin(x)$
$\implies k = \max_{x\in[0,\pi]} |-\sin(x)| = 1$, So now, we have$|E_M| \leq \frac{k\cdot\pi^3}{24\cdot4^2} = \frac{1\cdot\pi^3}{24\cdot4^2}$
$= \frac{\pi^3}{4^2\cdot24} = \frac{\pi^3}{384}$

(c) (4 points) Now find the exact error in your estimate from part (a).

SOLUTION:

Exact Error $= \int_0^{\pi} \sin(x)\, dx$ - $M_4 = \boxed{2 - M_4}$

3. (a) (10 points) Evaluate the following integral $\displaystyle\int_0^3 \frac{1}{(x-1)^{2/3}}\,dx$ and determine whether it converges or diverges.

SOLUTION:

This function is undefined when $x = 1$, so we need to split the problem into 2 integrals.

$$\int_0^3 \frac{1}{(x-1)^{2/3}}\,dx = \int_0^1 \frac{1}{(x-1)^{2/3}}\,dx + \int_1^3 \frac{1}{(x-1)^{2/3}}\,dx$$

$$\implies \lim_{b\to 1}\int_0^b \frac{1}{(x-1)^{2/3}}\,dx = 3$$

$$\implies \lim_{c\to 1}\int_c^3 \frac{1}{(x-1)^{2/3}}\,dx = 3\cdot 2^{2/3}$$

The two integrals on the right hand side both converge and add up to $\boxed{3[1+2^{\frac{2}{3}}]}$

(b) (10 points) Determine whether the following integral converges or diverges.

$$\int_1^\infty \frac{\sin^2(x)}{x^2}\,dx$$

SOLUTION:

$0 \le \sin^2(x) \le 1$ for all x so

$0 \le \frac{\sin^2(x)}{x^2} \le \frac{1}{x^2}$ for all $x \ge 1$

Since $\displaystyle\int_1^\infty \frac{1}{x^2}dx$ converges by the p-test, so does $\displaystyle\int_1^\infty \frac{\sin^2(x)}{x^2}dx$

4. Evaluate the following integrals. **Show all work!**

(a) (12 points) $\displaystyle\int \frac{\ln(x)\ln(\ln(x))}{x}\,dx$

SOLUTION:

u $= \ln(x)$

du $= \frac{1}{x}dx$

$\Longrightarrow \displaystyle\int u \cdot \ln(u)du \Longrightarrow$ u $= \ln(\text{u})$, du $= \dfrac{1}{u}du$, dv $= u$, v $= \dfrac{u^2}{2}$

$\Longrightarrow \dfrac{\ln(u)u^2}{2} - \displaystyle\int \frac{u}{2}du = \frac{u^2 \cdot \ln(u)}{2} - \frac{u^2}{4} + c$

$= \boxed{\dfrac{(\ln(x))^2 \cdot \ln(\ln(x))}{2} - \dfrac{(\ln(x))^2}{4} + c}$

(b) (12 points) $\displaystyle\int x^3\sqrt{4+x^2}\,dx$

SOLUTION:

x $= 2\tan(\theta), dx = 2sec^2\theta d\theta$

$\Longrightarrow \displaystyle\int 8\tan^3\theta\sqrt{4+4\tan^2\theta}2sec^2\theta d\theta$

$\Longrightarrow 32\displaystyle\int \tan^3\theta\sec^3\theta d\theta$

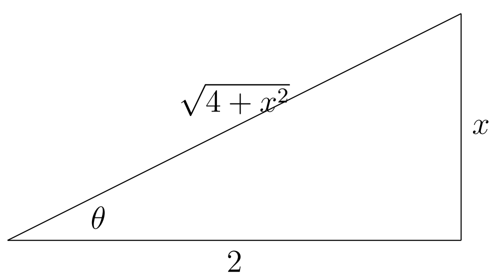
$= 32\displaystyle\int (\sec^2\theta - 1)\sec^2\theta\sec(\theta)\tan(\theta)d\theta$

$\Longrightarrow u = \sec\theta, du = \sec\theta\tan\theta d\theta$

$= 32\displaystyle\int (u^4 - u^2)du$

$= 32[\dfrac{u^5}{5} - \dfrac{u^3}{3}] + c$

$= 32[\dfrac{sec^5(\theta)}{5} - \dfrac{sec^3(\theta)}{3}] + c$



$= \boxed{32\left(\dfrac{\left(\frac{(\sqrt{4+x^2})}{2}\right)^5}{5} - \dfrac{\left(\frac{(\sqrt{4+x^2})}{2}\right)^3}{3} + c\right)}$

(c) (12 points) $\int \frac{3x^2}{x^2+1} dx$

SOLUTION:

$$\int \frac{3x^2}{x^2+1} dx = 3\int \frac{x^2}{x^2+1} dx$$

$$\implies \text{After long division} = 3\left(\int dx - \int \frac{1}{x^2+1} dx\right) = \boxed{3x - 3\tan^{-1}(x) + c}$$