1. Let \( \mathcal{R} \) be the region bounded by \( y = 2 - x \) and \( y = x^2 \)

   (a) (5 points) Sketch the region. Be sure to label all axes, curves, and intersection points.

   **SOLUTION:**

   ![Figure 1: Bounded Area Region](image)

   (b) (7 points) Set up the dx integral(s) (integral(s) with respect to \( x \)) which, if evaluated, would give the area of the region. **DO NOT EVALUATE.**

   **SOLUTION:**

   \[
   \text{Area} = \int_{-2}^{1} (2 - x - x^2) \, dx
   \]

   (c) (7 points) Set up the dy integral(s) (integral(s) with respect to \( y \)) which, if evaluated, would give the area of the region. **DO NOT EVALUATE.**

   **SOLUTION:**

   \[
   A = \int_{0}^{1} 2\sqrt{y} \, dy + \int_{1}^{4} (2 - y) - \sqrt{y} \, dy
   \]

   (d) (5 points) Find the area of the region \( \mathcal{R} \) from either (b) or (c).

   **SOLUTION:**

   Evaluation of either (b) or (c) will yield a result of \( \frac{9}{2} \)
2. Consider the curve $y = \sin(x)$.

(a) (8 points) Use the Midpoint Rule with $n = 4$ sub-intervals to estimate the integral of this curve from $x = 0$ to $x = \pi$.

**SOLUTION:**

Here, we know that $n = 4, a = 0, b = \pi$. Thus, we can say $\Delta x = \frac{(b-a)}{n} = \frac{\pi}{4}$. We now divide the interval $[0, 4\pi]$ into $n = 4$ sub-intervals and of the length $\Delta x = \frac{\pi}{4}$ with the following endpoints: $a = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi = b$.

Now we just evaluate the function at the midpoints of the sub-intervals.

$\begin{align*}
  f\left(\frac{x_0 + x_1}{2}\right) &= f\left(\frac{0 + \frac{\pi}{4}}{2}\right) = f\left(\frac{\pi}{8}\right) \\
  f\left(\frac{x_1 + x_2}{2}\right) &= f\left(\frac{\frac{\pi}{4} + \frac{\pi}{2}}{2}\right) = f\left(\frac{3\pi}{8}\right) \\
  f\left(\frac{x_2 + x_3}{2}\right) &= f\left(\frac{\frac{\pi}{2} + \frac{3\pi}{4}}{2}\right) = f\left(\frac{5\pi}{8}\right) \\
  f\left(\frac{x_3 + x_4}{2}\right) &= f\left(\frac{\frac{3\pi}{4} + \pi}{2}\right) = f\left(\frac{7\pi}{8}\right)
\end{align*}$

Therefore $M_4 = \frac{\pi}{4}[\sin\left(\frac{\pi}{8}\right) + \sin\left(\frac{3\pi}{8}\right) + \sin\left(\frac{5\pi}{8}\right) + \sin\left(\frac{7\pi}{8}\right)]$

(b) (8 points) Estimate the error in the calculation from part (a)

**SOLUTION:**

To estimate the error, we can do it with the bounding formula,

$$|E_M| \leq \frac{k(b-a)^3}{24n^2}$$

where $k$ is the maximum value of $|\frac{d^2y}{dx^2}|$ on $x \in [0, \pi]$.

$\therefore |E_M| \leq \frac{k(b-a)^3}{24n^2}$

$= \frac{b \pi^4}{24n^2}$

we can find $k$ now $\frac{d^2y}{dx^2} = -\sin(x)$

$
\implies k = \max_{x\in[0,\pi]} | - \sin(x) | = 1,

So now, we have $|E_M| \leq \frac{k \pi^3}{24n^2} = \frac{\pi^3}{24 \cdot 4^2} = \frac{\pi^3}{384}$

(c) (4 points) Now find the exact error in your estimate from part (a).

**SOLUTION:**

Exact Error $= \int_0^\pi \sin(x) \, dx - M_4 = [2 - M_4]$
3. (a) (10 points) Evaluate the following integral \( \int_0^3 \frac{1}{(x-1)^{2/3}} \, dx \) and determine whether it converges or diverges.

**SOLUTION:**

This function is undefined when \( x = 1 \), so we need to split the problem into 2 integrals.

\[
\int_0^3 \frac{1}{(x-1)^{2/3}} \, dx = \int_0^1 \frac{1}{(x-1)^{2/3}} \, dx + \int_1^3 \frac{1}{(x-1)^{2/3}} \, dx
\]

\[
\Rightarrow \lim_{b \to 1} \int_0^b \frac{1}{(x-1)^{2/3}} \, dx = 3
\]

\[
\Rightarrow \lim_{c \to 1} \int_c^3 \frac{1}{(x-1)^{2/3}} \, dx = 3 \cdot 2^{2/3}
\]

The two integrals on the right hand side both converge and add up to \( 3[1 + 2^{2/3}] \)

(b) (10 points) Determine whether the following integral converges or diverges.

\[
\int_1^\infty \frac{\sin^2(x)}{x^2} \, dx
\]

**SOLUTION:**

\[
0 \leq \sin^2(x) \leq 1 \quad \text{for all } x
\]

\[
0 \leq \frac{\sin^2(x)}{x^2} \leq \frac{1}{x^2} \quad \text{for all } x \geq 1
\]

Since \( \int_1^\infty \frac{1}{x^2} \, dx \) converges by the p-test, so does \( \int_1^\infty \frac{\sin^2(x)}{x^2} \, dx \)
4. Evaluate the following integrals. **Show all work!**

(a) (12 points) \( \int \frac{\ln(x)\ln(\ln(x))}{x} \, dx \)

**SOLUTION:**

\( u = \ln(x) \)
\( du = \frac{1}{x} \, dx \)

\[ \Rightarrow \int u \cdot \ln(u) \, du \quad \Rightarrow \quad u = \ln(u) \, , \, dv = u \, , \, v = \frac{u^2}{2} \]

\[ \Rightarrow \frac{\ln(u)u^2}{2} - \int \frac{u^2}{2} \, du = \frac{u^2 \cdot \ln(u)}{2} - \frac{u^2}{4} + c \]

\[ = \frac{(\ln(x))^2 \cdot \ln(\ln(x))}{2} - \frac{(\ln(x))^2}{4} + c \]

(b) (12 points) \( \int x^3\sqrt{4 + x^2} \, dx \)

**SOLUTION:**

\( x = 2\tan(\theta) , \, dx = 2\sec^2(\theta) \, d\theta \)

\[ \Rightarrow \int 8\tan^3(\theta)\sqrt{4 + 4\tan^2(\theta)}2\sec^2(\theta) \, d\theta \]

\[ \Rightarrow 32 \int \tan^3(\theta)\sec^3(\theta) \, d\theta \]

\[ = 32 \int (\sec^2(\theta) - 1)\sec^2(\theta)\sec(\theta)\tan(\theta) \, d\theta \]

\[ \Rightarrow \quad u = \sec(\theta) \, , \, du = \sec(\theta)\tan(\theta) \, d\theta \]

\[ = 32 \int (u^4 - u^2) \, du \]

\[ = 32\left[ \frac{u^5}{5} - \frac{u^3}{3} \right] + c \]

\[ = 32\left[ \frac{\sec^5(\theta)}{5} - \frac{\sec^3(\theta)}{3} \right] + c \]

\[ = 32 \left( \frac{\sqrt{4 + x^2}}{5} \right)^5 - \left( \frac{\sqrt{4 + x^2}}{3} \right)^3 + c \]
(c) (12 points) \[ \int \frac{3x^2}{x^2 + 1} \, dx \]

**SOLUTION:**

\[ \int \frac{3x^2}{x^2 + 1} \, dx = 3 \int \frac{x^2}{x^2 + 1} \, dx \]

\[ \Rightarrow \text{After long division} = 3 \left( \int dx - \int \frac{1}{x^2 + 1} \, dx \right) = 3x - 3\tan^{-1}(x) + c \]