1 Evaluate the following integrals. Be sure to simplify your answers.

(a) (14 points) $\int \ln(\sqrt{x})dx$

(b) (14 points) $\int \frac{x+8}{x^2+x-2}dx$

Solution: (a) Using integeration by parts with $u = \ln(x), du = \frac{1}{x}dx, dv = dx, v = x$

$$\implies \int \ln(\sqrt{x})dx = \frac{1}{2}\int \ln(x)dx = \frac{xln(x) - x}{2} + C$$

(b) Use integration by partial fractions to solve $\int \frac{x+8}{x^2+x-2}dx$

$$\implies \frac{x+8}{x^2+x-2} = \frac{x+8}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$
$$\implies \frac{x+8}{(x+2)(x-1)} = \frac{A(x+2)+B(x-2)}{(x-1)(x+2)}$$
$$\implies x+8 = A(x+2) + B(x-1)$$
$$\implies A = 3 \quad B = -2,$$
$$= \int \frac{3}{x-1} - \frac{2}{x+2}dx = 3\ln|x-1| - 2\ln|x+2| + C$$
$$= \ln\frac{|x-1|^3}{|x+2|^2} + C$$

2 Consider the region R in the first quadrant under the line $y = 2 - x$. Set up but **do not evaluate** integrals to find the following quantities.

(a) (12 points) The volume of the solid obtained by rotating the region R about the line $x = -1$ using the Disk/Washer Method.

(b) (12 points) The volume of the solid obtained by rotating the region R about the line $x = -1$ using the Cylindrical Shells Method.

Solution: (a) Since the line of rotation is parallel to the y-axis we must integrate with respect to dy. By the washer method we have

$$A = \int dA = \int \pi(R^2 - r^2)\,dy$$

Since the line of rotation is $x = -1$, $R = 1 + x = 1 + (2 - y) = 3 - y$, and $r = 1$. The range for y's in which the disks lie will be between 0 to 2. This gives the integral:

$$A = \pi\int_0^2 (3-y)^2 - 1\,dy$$

(b) Using the shell method we will integrate with respect to dx.

$$A = \int 2\pi rh\,dx$$

The radius of the shells will be $r = 1 + x$ while the height of each shell is given by $h = y = 2 - x$. The x range for the shells is from 0 to 2. Thus, the integral representing the volume is

$$A = 2\pi \int_0^2 (1+x)(2-x)\,dx$$

3 (10 points) Consider the curve defined by $x = \frac{1}{2}y^2$ on $0 \leq x \leq 2$. Set up but **do not evaluate** the surface area of the solid obtained by rotating the curve about the **x-axis**.

Solution: $\text{SA} = \int_0^2 2\pi\sqrt{2x}\sqrt{1 + \frac{1}{2x}}dx$

4 Does the sequence or series converge? If so, what does it converge to? Justify your answer and name any tests or theorems you use.

(a) (10 points) $a_n = \dfrac{\ln n}{\sqrt[3]{n}}$

(b) (10 points) $\displaystyle\sum_{n=1}^{\infty} \frac{5}{4^n}$

Solution: (a)

$$\begin{aligned}
\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\ln n}{\sqrt[3]{n}} &= \frac{\infty}{\infty} \\
&\overset{\text{L'H}}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{3}n^{-\frac{2}{3}}} \\
&= \lim_{n\to\infty} \frac{1}{n} 3n^{\frac{2}{3}} \\
&= \lim_{n\to\infty} \frac{3}{n^{\frac{1}{3}}} \\
&= 0
\end{aligned}$$

(b)

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{5}{4^n} &= 5 \cdot \frac{1}{4} \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \quad \text{(Converges by geometric series)} \\
&= 20\left(\frac{1}{1 - \frac{1}{4}}\right) \quad \text{(geometric series formula)} \\
&= \frac{5}{4} \cdot \frac{4}{3} = \frac{5}{3}
\end{aligned}$$

5 Consider the function $f(x) = \arctan(x)$.

(a) (12 points) Show that the Maclaurin series of $f(x)$ has a radius of convergence of 1 using a relevant calculation.

(b) (12 points) Find a power series representation for $g(x) = \frac{x}{2}\arctan(x^2)$ centered at 0.

Solution: (a) First, note that the Maclaurin series for $f(x)$ is

$$f(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}.$$

Applying the ratio test,

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{x^{2n+3}}{2n+3}\frac{2n+1}{x^{2n+1}}\right| \\ &= \lim_{n\to\infty}\frac{2n+3}{2n+1}|x|^2 \\ &= x^2.\end{aligned}$$

$x^2 < 1$ for $-1 < x < 1$; the radius of convergence is 1.

(b) Note that $g(x) = \frac{x}{2}f(x^2)$. Thus, using the Maclaurin series stated in part (a), we have

$$\begin{aligned}g(x) &= \frac{x}{2}\sum_{n=0}^{\infty}(-1)^n \frac{(x^2)^{2n+1}}{2n+1} \\ &= \sum_{n=0}^{\infty}(-1)^n \frac{x}{2}\frac{x^{4n+2}}{2n+1} \\ &= \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+3}}{4n+2}.\end{aligned}$$

6. Consider the parametric curve $\begin{matrix} x(t) = (t-1)^2 \\ y(t) = t - t^2 \end{matrix}$ where $0 \leq t \leq 1$.

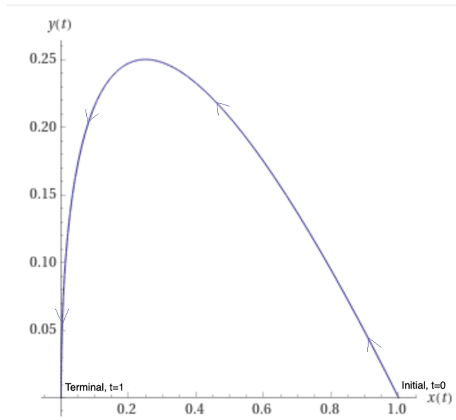
(a) (12 points) Sketch the parametric cure. Mark the initial and terminal points.

(b) (12 points) Set-up but **do not evaluate** the area under the curve using a parametric integral.

Solution: (a) Graph below:

Some tips for graphing this without using many test points: the initial point is $(x(0), y(0)) = (1, 0)$; the terminal point is $(x(1), y(1)) = (0, 0)$. Note also that $\frac{d}{dt}x(t) = 2(t-1) < 0$ for $t \in (0,1)$, so the x-coordinate is decreasing for the full duration of the trajectory.

Conversely, $\frac{d}{dt}y(t) = 1 - 2t$, which is positive for $t \in (0, 1/2)$, so the y-coordinate initially increases until $t = 1/2$, and negative for $t \in (1/2, 1)$, so the y-coordinate decreases after $t = 1/2$. This results in a local maximum of the curve occurring at $t = 1/2$: $(x(1/2), y(1/2)) = (1/4, 1/4)$.



(b) We observed above that $x(t)$ is strictly decreasing on the given time interval, so to obtain a positive area, we must integrate backward in time:

$$A = \int_1^0 y(t)x'(t)dt = \int_1^0 (t-t^2)(2(t-1))dt = 2\int_0^1 (t-t^2)(t-1)dt = -2\int_0^1 (t-t^2)(t-1)dt.$$

7 Consider the polar curve $r = \dfrac{2|\theta|}{\pi}$ for $-\pi \leq \theta \leq \pi$.

(a) (10 points) Set-up but **do note evaluate** an integral to find the length of the curve.

(b) (10 points) Set-up but **do note evaluate** an integral to find the area bounded by the y-axis and $r(\theta)$, where $x \geq 0$.

Solution: (a) The length of a polar curve is given by

$$\int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

Since $r(\theta)$ is not differentiable at $\theta = 0$, we split the integral into two regions: for $-\pi \leq \theta\pi$, $\frac{dr}{d\theta} = \frac{d}{d\theta}(-2\theta/\pi) = -2/\pi$. For $0 \leq \theta \leq \pi$, $\frac{dr}{d\theta} = \frac{d}{d\theta}2\theta/\pi$. The length of the curve is, therefore,

$$
\begin{aligned}
L &= \int_{-\pi}^{0} \sqrt{\left(\frac{2\theta}{\pi}\right)^2 + \left(-\frac{2}{\pi}\right)^2} d\theta + \int_{0}^{\pi} \sqrt{\left(\frac{2\theta}{\pi}\right)^2 + \left(\frac{2}{\pi}\right)^2} \\
&= \int_{-\pi}^{0} \sqrt{\left(\frac{2\theta}{\pi}\right)^2 + \left(\frac{2}{\pi}\right)^2} d\theta + \int_{0}^{\pi} \sqrt{\left(\frac{2\theta}{\pi}\right)^2 + \left(\frac{2}{\pi}\right)^2} \\
&= \int_{-\pi}^{\pi} \sqrt{\left(\frac{2\theta}{\pi}\right)^2 + \left(\frac{2}{\pi}\right)^2} d\theta \\
&= \frac{2}{\pi} \int_{-\pi}^{\pi} \sqrt{\theta^2 + 1} d\theta \\
\text{or } &= \frac{4}{\pi} \int_{0}^{\pi} \sqrt{\theta^2 + 1} d\theta
\end{aligned}
$$

where the final equality makes use of the fact that the integrand is even.

(b) The polar area integral is given by

$$
A = \frac{1}{2} \int_{\alpha}^{\beta} (r(\theta))^2 d\theta.
$$

Here, $\alpha = -\pi/2$, $\beta = \pi/2$, so

$$
A = \frac{1}{2} \int_{-\pi/2}^{\pi/2} \left(\frac{2|\theta|}{\pi}\right)^2 d\theta = \frac{2}{\pi^2} \int_{-\pi/2}^{\pi/2} \theta^2 d\theta \overset{\text{or}}{=} \frac{4}{\pi^2} \int_{0}^{\pi/2} \theta^2 d\theta,
$$

where the final equality makes use of the fact that the integrand is even.