1. Determine if the series converge or diverge. Be sure to fully justify your answer and state what test that you used.

   (a) (10 points) $\displaystyle\sum_{n=1}^{\infty} \frac{n-4}{n^3+n}$.

   (b) (10 points) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n[3\cdot 6\cdot 9\cdots(3n)]}{(n+2)!}$

   (c) (10 points) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2-\cos^2(n)}$.

Solution: (a) This series **converges** by the **direct comparison test**. For $n > 4$,

$$0 < \frac{n-4}{n^3+n} < \frac{n}{n^3+n} < \frac{n}{n^3} = \frac{1}{n^2}.$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series, we conclude that $\displaystyle\sum_{n=1}^{\infty} \frac{n-4}{n^3+n}$ converges.

(b) This series **diverges** by the **ratio test.** Let $a_n = \dfrac{(-1)^n[3\cdot 6\cdot 9\cdots(3n)]}{(n+2)!}$. Then

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{[3\cdot 6\cdot 9\cdots(3n)\cdot(3n+3)]}{(n+3)!}\frac{(n+2)!}{[3\cdot 6\cdot 9\cdots(3n)]}\right| \\ &= \lim_{n\to\infty}\frac{(3n+3)}{n+3} \\ &= 3 > 1.\end{aligned}$$

The ratio test concludes that the series diverges.

(c) The series **diverges** by the **limit comparison test**. Observe that the harmonic series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ diverges by the p-test ($p = 1$). Moreover,

$$\lim_{n\to\infty}\frac{1/n}{n/(n^2-\cos^2(n))} = \lim_{n\to\infty}\frac{n^2-\cos^2(n))}{n^2} = \lim_{n\to\infty} 1 - \frac{\cos^2(n)}{n^2} = 1 \in (0,\infty).$$

By the limit comparison test, $\displaystyle\sum_{n=1}^{\infty}\frac{n}{n^2-\cos^2(n)}$ diverges because $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

2. Determine the interval of convergence and the radius of convergence for the following power series.

   (a) (15 points) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n n}{4^n}(x+3)^n$

   (b) (15 points) $\displaystyle\sum_{n=1}^{\infty}\frac{(2x-1)^n}{\sqrt{n}3^n}$

Solution: (a) **Apply the ratio test** with $a_n = \frac{(-1)^n n}{4^n}(x+3)^n$:

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(n+1)(x+3)^{n+1}}{4^{(n+1)}}\frac{4^n}{n(x+3)^n}\right| \\
&= \lim_{n\to\infty}\frac{1}{4}\left(\frac{n+1}{n}\right)|x+3| \\
&= \frac{|x+3|}{4}
\end{aligned}$$

We have absolute convergence when $\frac{1}{4}|x+3| < 1$, which holds for $-7 < x < 1$. **The radius of convergence is 4**.

To determine the interval of convergence, we check the endpoints. For $x = -7$ we obtain the series

$$\sum_{n=1}^{\infty}\frac{(-1)^n n}{4^n}(-4)^n = \sum_{n=1}^{\infty} n,$$

which diverges (test for divergence, $\lim_{n\to\infty} n = \infty \neq 0$).

For $x = 1$ we obtain the series

$$\sum_{n=1}^{\infty}\frac{(-1)^n n}{4^n}4^n = \sum_{n=1}^{\infty}(-1)^n n,$$

which also diverges by the test for divergence since $\lim_{n\to\infty}(-1)^n n$ does not exist.

The interval of convergence, therefore, is (-7,1) and the radius of convergence is 4.

(b) **Apply the ratio test** with $a_n = \frac{(2x-1)^n}{\sqrt{n}3^n}$:

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(2x-1)^{n+1}}{\sqrt{n+1}3^{n+1}}\frac{\sqrt{n}3^n}{(2x-1)^n}\right| \\
&= \lim_{n\to\infty}\frac{1}{3}\sqrt{\frac{n+1}{n}}|2x-1| \\
&= \frac{1}{3}|2x-1|
\end{aligned}$$

Absolute convergence is guaranteed where $\frac{1}{3}|2x-1| < 1$, or $-1 < x < 2$. Checking the $x = -1$ endpoint,

$$\sum_{n=1}^{\infty}\frac{(-3)^n}{\sqrt{n}3^n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}.$$

This is a converges via the **alternating series test**, as $\frac{1}{\sqrt{n}}$ is decreasing and $\lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0$.

At the $x = 2$ endpoint, we obtain the series

$$\sum_{n=1}^{\infty} \frac{3^n}{\sqrt{n}3^n} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}},$$

which is a **divergent p-series** ($p = 1/2 \leq 1$).

The interval of convergence is [-1,2); the radius of convergence is 1.5.

3. Suppose the series $B = \sum_{n=0}^{\infty} d_n(x-7)^n$ has a radius of convergence of 3. Determine if the following series converge, diverge, or if there is not enough information to determine the series' behavior.

Solution: We are given the series $B = \sum_{n=0}^{\infty} d_n(x-7)^n$ and told its radius of convergence is 3. Notice that B is not just a power series but a power series centered at $x = 7$. Thus, if we have a radius of convergence of 3, then we assume a plausible interval of convergence to be $(7-3, 7+3) = (4, 10)$. **Note: we do NOT know anything about the endpoints and so this interval is left open**.

(a) We are given the series $\sum_{n=0}^{\infty} d_n 3^n$. We can put this into a form that is similar to B.

$$\sum_{n=0}^{\infty} d_n(3)^n = \sum_{n=0}^{\infty} d_n(10-7)^n$$

$$\implies x = 10$$

$$\implies x \notin (4, 10)$$

(we do not know anything about the convergence at the endpoints!)

$\therefore$ $\boxed{\text{The answer is INCONCLUSIVE}}$

(b) Similar to (a), we are given the series $\sum_{n=0}^{\infty} d_n(-1)^n$. We can put this into a form that is similar to B again.

$$\sum_{n=0}^{\infty} d_n(-1)^n = \sum_{n=0}^{\infty} d_n(6-7)^n$$

$$\implies x = 6$$

$$\implies x \in (4, 10)$$

(x is strictly within our interval of convergence)

$\therefore$ $\boxed{\text{The answer is CONVERGES}}$

4.

(a) (4 points) Write down a power series representation for $\frac{1}{1+x}$.

SOLUTION: $\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n$

(b) (8 points) Use your answer from part (a) to find a power series representation for $f(x) = \ln(1+x)$

SOLUTION: Inside the interval of convergence $x \in (-1,1)$ we can integrate the series term by term:

$$\int_0^x \frac{dt}{1+t} = \sum_{n=0}^{\infty} \int_0^x (-1)^n t^n dt$$

and obtain a series with a radius of convergence of $R = 1$

$$\ln(1+x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}$$

5.

(a) Write down the definition for a Maclaurin for a general function $f(x)$. Assume $f(x)$ has derivatives of all orders.

SOLUTION:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

(b) Use the definition from part (a) to find the Maclaurin series representation for

$$f(x) = \cosh(x) = \frac{e^x + e^{-x}}{2}$$

SOLUTION:

First compute derivatives:

$$f(x) = \cosh(x)$$

$$f'(x) = \frac{d}{dx}\cosh(x) = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x - e^{-x}}{2} = \sinh(x)$$

$$f''(x) = \frac{d}{dx}\sinh(x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh(x)$$

$$f'''(x) = \frac{d}{dx}\cosh(x) = \sinh(x)$$

$$\vdots$$

It's clear from the above computations that we have the following for the n^{th} derivative of $f(x)$:

$$f^{(n)}(x) = \begin{cases} \cosh(x) & n \text{ even} \\ \sinh(x) & n \text{ odd} \end{cases}$$

Evaluating at 0 gives

$$f^{(n)}(0) = \begin{cases} 1 & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

Using this in our formula from (a) and reindexing the series to drop the odd terms gives

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}$$