1. Determine if the series converge or diverge. Be sure to fully justify your answer and state what test that you used.

(a) (10 points) \( \sum_{n=1}^{\infty} \frac{n-4}{n^3+n} \).

(b) (10 points) \( \sum_{n=1}^{\infty} \frac{(-1)^n[3 \cdot 6 \cdot 9 \cdots (3n)]}{(n+2)!} \).

(c) (10 points) \( \sum_{n=1}^{\infty} \frac{n}{n^2-\cos^2(n)} \).

Solution: (a) This series converges by the direct comparison test. For \( n > 4 \),

\[
0 < \frac{n-4}{n^3+n} < \frac{n}{n^3+n} < \frac{n}{n^3} = \frac{1}{n^2}.
\]

Since \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) is a convergent p-series, we conclude that \( \sum_{n=1}^{\infty} \frac{n-4}{n^3+n} \) converges.

(b) This series diverges by the ratio test. Let \( a_n = \frac{(-1)^n[3 \cdot 6 \cdot 9 \cdots (3n)]}{(n+2)!} \).

Then

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{3 \cdot 6 \cdot 9 \cdots (3n) \cdot (3n+3)}{(n+3)!} \frac{(n+2)!}{3 \cdot 6 \cdot 9 \cdots (3n)} \right| = \lim_{n \to \infty} \frac{(3n+3)}{n+3} = 3 > 1.
\]

The ratio test concludes that the series diverges.

(c) The series diverges by the limit comparison test. Observe that the harmonic series \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges by the p-test \( (p = 1) \). Moreover,

\[
\lim_{n \to \infty} \frac{1/n}{n/(n^2-\cos^2(n))} = \lim_{n \to \infty} \frac{n^2-\cos^2(n)}{n^2} = \lim_{n \to \infty} 1 - \frac{\cos^2(n)}{n^2} = 1 \in (0, \infty).
\]

By the limit comparison test, \( \sum_{n=1}^{\infty} \frac{n}{n^2-\cos^2(n)} \) diverges because \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges.

2. Determine the interval of convergence and the radius of convergence for the following power series.

(a) (15 points) \( \sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n} (x+3)^n \).

(b) (15 points) \( \sum_{n=1}^{\infty} \frac{(2x-1)^n}{\sqrt{n}3^n} \).
Solution: (a) **Apply the ratio test** with \( a_n = \frac{(-1)^n n}{4^n} (x + 3)^n \):

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(n + 1)(x + 3)^{n+1}}{4^{n+1}} \cdot \frac{4^n}{n(x + 3)^n} \right| \\
= \lim_{n \to \infty} \frac{1}{4} \left( \frac{n + 1}{n} \right) |x + 3| \\
= \frac{|x + 3|}{4}
\]

We have absolute convergence when \( \frac{1}{4} |x + 3| < 1 \), which holds for \(-7 < x < 1\). **The radius of convergence is 4.**

To determine the interval of convergence, we check the endpoints. For \( x = -7 \) we obtain the series

\[
\sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n} (-4)^n = \sum_{n=1}^{\infty} n,
\]

which diverges (test for divergence, \( \lim_{n \to \infty} n = \infty \neq 0 \)).

For \( x = 1 \) we obtain the series

\[
\sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n} 4^n = \sum_{n=1}^{\infty} (-1)^n n,
\]

which also diverges by the test for divergence since \( \lim_{n \to \infty} (-1)^n n \) does not exist.

**The interval of convergence, therefore, is \((-7, 1)\) and the radius of convergence is 4.**

(b) **Apply the ratio test** with \( a_n = \frac{(2x-1)^n}{\sqrt{n} 3^n} \):

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(2x - 1)^{n+1} \sqrt{n} 3^n}{\sqrt{n + 1} 3^{n+1} (2x - 1)^n} \right| \\
= \lim_{n \to \infty} \frac{1}{3} \sqrt{\frac{n + 1}{n}} |2x - 1| \\
= \frac{1}{3} |2x - 1|
\]

Absolute convergence is guaranteed where \( \frac{1}{3} |2x - 1| < 1 \), or \(-1 < x < 2\). Checking the \( x = -1 \) endpoint,

\[
\sum_{n=1}^{\infty} \frac{(-3)^n}{\sqrt{n} 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}.
\]

This is a converges via the **alternating series test**, as \( \frac{1}{\sqrt{n}} \) is decreasing and \( \lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0 \).
At the $x = 2$ endpoint, we obtain the series

$$
\sum_{n=1}^{\infty} \frac{3^n}{\sqrt{n}3^n} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}},
$$

which is a divergent $p$-series ($p = 1/2 \leq 1$).

The interval of convergence is $[-1, 2)$; the radius of convergence is 1.5.

3. Suppose the series $B = \sum_{n=0}^{\infty} d_n(x - 7)^n$ has a radius of convergence of 3. Determine if the following series converge, diverge, or if there is not enough information to determine the series’ behavior.

Solution: We are given the series $B = \sum_{n=0}^{\infty} d_n(x - 7)^n$ and told its radius of convergence is 3. Notice that $B$ is not just a power series but a power series centered at $x = 7$. Thus, if we have a radius of convergence of 3, then we assume a plausible interval of convergence to be $(7 - 3, 7 + 3) = (4, 10)$. **Note:** we do NOT know anything about the endpoints and so this interval is left open.

(a) We are given the series $\sum_{n=0}^{\infty} d_n3^n$. We can put this into a form that is similar to $B$.

$$
\sum_{n=0}^{\infty} d_n(3)^n = \sum_{n=0}^{\infty} d_n(10 - 7)^n
$$

$$
\implies x = 10
$$

$$
\implies x \notin (4, 10)
$$

(we do not know anything about the convergence at the endpoints!)

$: [The answer is INCONCLUSIVE]$

(b) Similar to (a), we are given the series $\sum_{n=0}^{\infty} d_n(-1)^n$. We can put this into a form that is similar to $B$ again.

$$
\sum_{n=0}^{\infty} d_n(-1)^n = \sum_{n=0}^{\infty} d_n(6 - 7)^n
$$

$$
\implies x = 6
$$

$$
\implies x \in (4, 10)
$$

($x$ is strictly within our interval of convergence)

$: [The answer is CONVERGES]$
4. 
(a) (4 points) Write down a power series representation for $\frac{1}{1+x}$.

**SOLUTION:**

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

(b) (8 points) Use your answer from part (a) to find a power series representation for $f(x) = \ln(1+x)$

**SOLUTION:** Inside the interval of convergence $x \in (-1, 1)$ we can integrate the series term by term:

$$\int_0^x \frac{dt}{1+t} = \sum_{n=0}^{\infty} \int_0^x (-1)^n t^n dt$$

and obtain a series with a radius of convergence of $R = 1$

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

5. 
(a) Write down the definition for a Maclaurin for a general function $f(x)$. Assume $f(x)$ has derivatives of all orders.

**SOLUTION:**

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

(b) Use the definition from part (a) to find the Maclaurin series representation for

$$f(x) = \cosh(x) = \frac{e^x + e^{-x}}{2}$$

**SOLUTION:**

First compute derivatives:

$$f(x) = \cosh(x)$$

$$f'(x) = \frac{d}{dx} \cosh(x) = \frac{d}{dx} \left( \frac{e^x + e^{-x}}{2} \right) = \frac{e^x - e^{-x}}{2} = \sinh(x)$$

$$f''(x) = \frac{d}{dx} \sinh(x) = \frac{d}{dx} \left( \frac{e^x - e^{-x}}{2} \right) = \frac{e^x + e^{-x}}{2} = \cosh(x)$$

$$f'''(x) = \frac{d}{dx} \cosh(x) = \sinh(x)$$

$$\vdots$$
It’s clear from the above computations that we have the following for the $n^{th}$ derivative of $f(x)$:

$$f^{(n)}(x) = \begin{cases} \cosh(x) & n \text{ even} \\ \sinh(x) & n \text{ odd} \end{cases}$$

Evaluating at 0 gives

$$f^{(n)}(0) = \begin{cases} 1 & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

Using this in our formula from (a) and reindexing the series to drop the odd terms gives

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}$$