1. Let $\mathcal{R}$ be the region bounded by the curve $y = \sqrt{x}$, the line $x = 4$, and the x-axis. The region $\mathcal{R}$ is rotated around the line $x = 6$ to form a solid.

   (a) (12 points) Set up an integral(s) for the volume of this solid using the Method of Cylindrical Shells. **EVALUATE THE INTEGRAL.**

   SOLUTION:

$$\int_0^4 2\pi(6-x)(\sqrt{x})dx = \int_0^4 12\pi\sqrt{x} - 2\pi x^{\frac{3}{2}}dx$$
$$= -\frac{4\pi(x-10)^{\frac{3}{2}}}{5}\Bigg|_0^4$$
$$= \frac{192\pi}{5}$$

   (b) (8 points) Set up an integral(s) for the volume of this solid using the Disk/Washer Method. **DO NOT EVALUATE THE INTEGRAL.**

   SOLUTION:

$$V_{washer} = \int_0^2 \pi[(6-y^2)^2 - 2^2]dy$$

2. (15 points) Consider the curve defined by $x = \frac{1}{2}y^2$ on $0 \le x \le 4$. Set up but **DO NOT EVALUATE** the surface area of the solid obtained by rotating the curve about the <u>x**-axis**</u>.

   SOLUTION:

$$S = \int_a^b 2\pi f(x)\sqrt{1 + f'(x)^2}\, dx$$

or

$$S = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

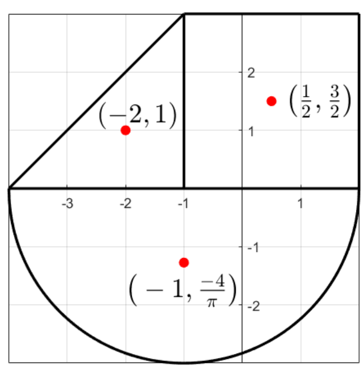
Either formula will work. In regards to dx, $y = \sqrt{2x}$ (we can use the top half of the curve since rotation about the x-axis will overlap with the bottom).

$$\frac{dy}{dx} = \frac{1}{\sqrt{2x}}$$
$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{2x}$$
$$S = 2\pi \int_0^4 \sqrt{2x}\sqrt{1 + \frac{1}{2x}}\, dx$$
$$= 2\pi \int_0^4 \sqrt{2x+1}\, dx$$

In regards to dy:

$$\frac{dx}{dy} = y$$
$$\left(\frac{dx}{dy}\right)^2 = y^2$$
$$S = 2\pi \int_0^{2\sqrt{2}} y\sqrt{1+y^2}\, dy$$

3. (15 points) Consider the region of uniform density $\rho = 1$ composed of a half circle, a square, and an isosceles triangle. Find the centroid for the region (the centroids of each smaller region are given).



SOLUTION:

Triangle:

$$m_T = \frac{1}{2}3(3)\rho = \frac{9\rho}{2} = \frac{9}{2}$$
$$(M_x)_T = m_T Y_T = \frac{9\rho}{2}(1) = \frac{9\rho}{2} = \frac{9}{2}$$
$$(M_y)_T = m_T X_T = \frac{9\rho}{2}(-2) = \frac{-18\rho}{2} = -9$$

Square:

$$m_s = (3)(3)\rho = 9\rho = 9$$
$$(M_x)_s = m_s Y_s = (9\rho)(\frac{3}{2}) = \frac{27\rho}{2} = \frac{27}{2}$$
$$(M_y)_s = m_s X_s = (9\rho)(\frac{1}{2}) = \frac{9\rho}{2} = \frac{9}{2}$$

Circle:

$$m_c = \frac{\pi}{2}3^2\rho = \frac{9\pi\rho}{2} = \frac{9\pi}{2}$$
$$(M_x)_c = m_c Y_c = (\frac{9\pi\rho}{2})(\frac{-4}{\pi}) = -18\rho = -18$$
$$(M_y)_c = m_c X_c = (\frac{9\pi\rho}{2})(-1) = -\frac{9\pi\rho}{2} = \frac{-9\pi}{2}$$

Total Mass

$$m = m_T + m_s + m_c = \frac{9}{2} + 9 + \frac{9\pi}{2} = \frac{27 + 9\pi}{2}$$

$$\implies M_x = (M_x)_T + (M_x)_s + (M_x)_c = (\frac{9}{2}) + (\frac{27}{2}) + (-18) = 0$$
$$\implies M_y = (M_y)_T + (M_y)_s + (M_y)_c = -9 + \frac{9}{2} + \frac{-9\pi}{2} = \frac{-9 - 9\pi}{2}$$
$$\implies \bar{x} = \frac{M_y}{m} = \frac{\frac{-9-9\pi}{2}}{\frac{27+9\pi}{2}} = \frac{-1 - \pi}{3 + \pi}$$
$$\implies \bar{y} = \frac{M_x}{m} = \frac{0}{\frac{27+9\pi}{2}} = 0$$

The centroid exists at $(\frac{-1-\pi}{3+\pi}, 0)$

4. For each of the following, determine whether the sequence converges or diverges. Explain your work in each case.

   (a) (6 points) $a_n = \ln(2n + 1) - \ln(n)$

   SOLUTION:

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \ln(2n + 1) - \ln(n) \\
&= \lim_{n\to\infty} \ln\left(\frac{2n + 1}{n}\right) \\
&= \ln\left(\lim_{n\to\infty} \frac{2n + 1}{n}\right) \\
&= \ln(2)
\end{aligned}$$

(b) (6 points) $c_n = \cos(n\pi/4)$

SOLUTION:

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} \cos(n\pi/4)$$

Note that the sequence will repeat the following values:

$$\begin{aligned}
c_0 &= \cos(0) = 1 \\
c_1 &= \cos(\pi/4) = \frac{\sqrt{2}}{2} \\
c_2 &= \cos(\pi/2) = 0 \\
c_3 &= -\frac{\sqrt{2}}{2} \\
c_4 &= -1 \\
c_5 &= -\frac{\sqrt{2}}{2} \\
c_6 &= 0 \\
c_7 &= \frac{\sqrt{2}}{2}
\end{aligned}$$

Since cosine is cyclic, this sequence will never get closer to a single value for large n. Therefore it diverges. Alternatively, the difference between any two consecutive terms $|c_n - c_{n+1}|$ will be larger than $\min(|1 - \frac{\sqrt{2}}{2}|, |0 - \frac{\sqrt{2}}{2}|) = 1 - \frac{\sqrt{2}}{2}$. i.e. since the difference between consecutive terms doesn't tend toward 0, the sequence will diverge.

(c) (6 points) $a_n = \frac{2^n}{5n^2}$

SOLUTION:

Considering the function $f(x) = \frac{2^x}{5x^2}$ we have that $f(n) = a)n$ for natural numbers, n. By L'Hopital's rule we compute:

$$\lim_{n\to\infty} f(x) = \lim_{n\to\infty} \frac{2^x}{5x^2} = \frac{\infty}{\infty}$$
$$\stackrel{\text{L'H}}{=} \lim_{n\to\infty} \frac{2^x \ln 2}{10x} = \frac{\infty}{\infty}$$
$$\stackrel{\text{L'H}}{=} \lim_{n\to\infty} \frac{2^x (\ln 2)^2}{10} = \infty$$

Moreover, $\frac{d}{dx}\frac{2^x}{5x^2} = \frac{(5x^2)(2^x \ln 2)-(2^x)(10x)}{25x^4} = \frac{5x2^x(x\ln 2-2)}{25x^4} > 0$ for $x \geq 2$. Since f is monotonically increasing for large enough values of x and since $f(n) = a_n$ on integer values, we must have that $\lim_{n\to\infty} a_n = \infty$ i.e. a_n diverges.

(d) (6 points) $b_n = \frac{3n+1}{(3n+1)!}$

SOLUTION:

$$\begin{aligned}
\lim_{n\to\infty} b_n &= \lim_{n\to\infty} \frac{3n+1}{(3n+1)!} \\
&= \lim_{n\to\infty} \frac{3n+1}{(3n+1)(3n)!} \\
&= \lim_{n\to\infty} \frac{1}{(3n)!} \\
&= 0
\end{aligned}$$

5. (6 points) If $d_1 = 5$ and $d_{n+1} = 2d_n$, find a closed form expression for d_n and then determine whether the sequence converges or diverges.
SOLUTION:
Writing out a few terms, we have

$$\begin{aligned} d_1 &= 5 \\ d_2 &= 2d_1 = 5 \cdot 2 \\ d_3 &= 2(d_2) = 2(2)(5) = 5 \cdot 2^2 \\ d_4 &= 2(d_3) = 2(2^2)(5) = 5 \cdot 2^3 \end{aligned}$$

We observe the pattern $d_n = 5 \cdot 2^{n-1}$. Indeed, this proposed sequence satisfies $d_1 = 5 \cdot 2^0 = 5$ and $d_{n+1}/d_n = \dfrac{5 \cdot 2^n}{5 \cdot 2^{n-1}} = 2 \Leftrightarrow d_{n+1} = 2d_n$.
This sequence diverges, since $\lim\limits_{x \to \infty} 5 \cdot 2^{x-1} = +\infty$.

6. (20 points) Solve the differential equation $\dfrac{dy}{dx} = \dfrac{2y}{x^2 - 1}$ given the initial condition $y(0) = 4$.
SOLUTION:

We are given the following differential equation and initial value,

$$\frac{dy}{dx} = \frac{2y}{x^2 - 1}; \;\; y(0) = 4 \tag{1}$$

First, note that we can put it in the following form

$$\begin{aligned} \frac{dy}{dx} &= \frac{2y}{x^2 - 1} \\ &= 2y \cdot \frac{1}{x^2 - 1} \\ &= g(y) \cdot f(x) \end{aligned}$$

Since we can decompose it like so, we know it must be *separable* and can attempt to solve the equation like so.

$$
\begin{aligned}
\implies \frac{dy}{dx} &= \frac{2y}{x^2-1} \\
\frac{dy}{2y} &= \frac{1}{x^2-1}dx \\
\int \frac{1}{2y}\,dy &= \int \frac{1}{x^2-1}\,dx \\
\frac{1}{2}\int \frac{1}{y}\,dy &= \int \frac{1}{(x+1)(x-1)}\,dx \\
\frac{1}{2}\ln|y| + C_1 &= \int \frac{1}{(x+1)(x-1)}\,dx
\end{aligned}
$$

Above, we have solved the left hand side but the right hand side remains to be evaluated. An initial idea of partial fraction decomposition seems to come to mind. Notice that the power of the numerator ($x^0 = 1$) is less than the highest power of the denominator (x^2). Thus, we do *not* have to do long division and can immediately start setting up partial fraction decomposition.

Doing so, we note that each quantity ($x+1$ and $x-1$) is not repeated (if it was repeated we would have something like $(x+1)^n$ or $(x-1)^m$, where n and m are integers greater than 1. We also see that the power of each quantity is to the first order((x^1+1) and (x^1-1)). Thus, we can set it up as the following

$$
\begin{aligned}
\frac{1}{(x+1)(x-1)} &= \frac{A}{x+1} + \frac{B}{x-1} \\
\implies 1 &= A(x-1) + B(x+1) \\
1 &= (A+B)x + (B-A)
\end{aligned}
$$

Now, we setup our equations to solve...

$$
\begin{aligned}
&A + B = 0 \implies A = -B \\
&B - A = 1 \implies B - (-B) = 1 \implies B = \frac{1}{2} \\
\implies &A = -\frac{1}{2} \text{ and } B = \frac{1}{2}
\end{aligned}
$$

Now, we plug our A and B in to find the value of the integral

$$\begin{aligned}\int \frac{1}{(x+1)(x-1)} &= \int \frac{-\frac{1}{2}}{x+1} + \frac{\frac{1}{2}}{x-1}\,dx \\ &= -\frac{1}{2}\int \frac{1}{x+1}\,dx + \frac{1}{2}\int \frac{1}{x-1}\,dx \\ &= -\frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + C_2\end{aligned}$$

Putting it all together (both sides of the equation from our problem), we have

$$\frac{1}{2}\ln|y| + C_1 = \frac{1}{2}\left[\ln|x-1| - \ln|x+1|\right] + C_2$$

Now, let $C_3 = C_2 - C_1$, $C_4 = 2C_3$, and $C_5 = e^{C_4}$. Then we have,

$$\begin{aligned}\implies \frac{1}{2}\ln|y| &= \frac{1}{2}\left[\ln|\frac{x-1}{x+1}|\right] + C_3 \\ \ln|y| &= \ln|\frac{x-1}{x+1}| + C_4 \\ y &= C_5 \cdot \frac{x-1}{x+1}\end{aligned}$$

Now, we apply the initial value of $y(0) = 4$ into our problem.

$$4 = y(0) \implies 4 = \frac{0-1}{0+1}\cdot C_5 \implies C_5 = -4$$

$$\implies \boxed{y = 4\left(\frac{1-x}{x+1}\right)}$$

— END —