1. Let $\mathcal{R}$ be the region in the first quadrant bounded by the curves $y = 2 + x^2$ on the right, $y = 5$ on top, and $y = 3x + 2$ on the left.

   (a) (5 points) Sketch the region. Be sure to label all axes, curves, and intersection points

   SOLUTION:

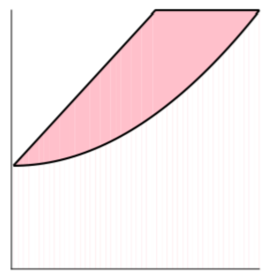


Figure 1: Bounded Area Region

   (b) (7 points) Set up the dx integral(s) (integral(s) with respect to x) which, if evaluated, would give the area of the region. **DO NOT EVALUATE**.

   SOLUTION:

$$\text{Area}=\int_0^1 (3x+2)-(x^2+2)\,dx+\int_1^{\sqrt{3}} 5-(x^2+2)\,dx$$

   (c) (7 points) Set up the dy integral(s) (integral(s) with respect to y) which, if evaluated, would give the area of the region. **DO NOT EVALUATE**.

   SOLUTION:

$$\text{Area}=\int_2^5 \sqrt{y-2}-\frac{y-2}{3}\,dy$$

   (d) (5 points) Find the area of the region $\mathcal{R}$ from either (b) or (c).

   SOLUTION:
   Evaluation of either (b) or (c) will yield a result of $2\sqrt{3}-\frac{3}{2}$.

2. Consider the curve $y = 2x^3$.

   (a) (10 points) Use the Trapezoid Rule with $n = 4$ sub-intervals to estimate the integral of this curve from $x = 1$ to $x = 4$. Leave your answer unsimplified.

   SOLUTION:

   Here, we know that $n = 4, a = 1$, and $b = 4$. Thus, we can say $\Delta x = \frac{(b-a)}{n} = \frac{3}{4}$. Using this, we can find each of our sub-intervals: $\{[1, \frac{7}{4}], [\frac{7}{4}, \frac{10}{4}], [\frac{10}{4}, \frac{13}{4}], [\frac{13}{4}, \frac{16}{4}]\}$.

   Now, recall that the area of each trapezoid is just the width times the average of the two endpoints for each interval, and so if we define a function for y, $f : x \mapsto 2x^2$, then

$$T_4 = \Delta x \left[\frac{f(1)+f(7/4)}{2} + \frac{f(7/4)+f(10/4)}{2} + \frac{f(10/4)+f(13/4)}{2} + \frac{f(13/4)+f(16/4)}{2}\right]$$

   or equivalently

$$= \frac{3}{8}\left[f(1) + 2 \cdot f(7/4) + 2 \cdot f(10/4) + 2 \cdot f(13/4) + f(16/4)\right]$$
$$= \boxed{\frac{3}{4}\left[1 + 2\left(\frac{7}{4}\right)^3 + 2\left(\frac{10}{4}\right)^3 + 2\left(\frac{13}{4}\right)^3 + 4^3\right]}$$

   (b) (10 points) Estimate $|E_T|$, the error in the trapezoidal approximation from part (a).

   SOLUTION:

   To estimate the error, we can do it with the bounding formula,

$$|E_T| \leq \frac{k(b-a)^3}{12n^2}$$

   where k is an upper bound for $|\frac{d^2y}{dx^2}|$ on $x \in [1, 4]$.

$$|E_T| \leq \frac{k(b-a)^3}{12n^2} = \frac{k \cdot 3^3}{12 \cdot 4^2}$$

   We can find a value for k now,

$$\frac{d^2y}{dx^2} = 12x \implies \text{let } k = \max_{x\in[1,4]} |12x| = 48.$$

   So now, we have

$$|E_T| \leq \frac{k \cdot 3^3}{12 \cdot 4^2} = \frac{48 \cdot 3^3}{12 \cdot 4^2} = \frac{3^3}{4} = \boxed{\frac{27}{4}}.$$

3. (a) (12 points) Does the improper integral $\displaystyle\int_1^2 \frac{1}{\sqrt{x-1}}dx$ converge? If yes, to what?

SOLUTION:
The lower bound of this integral makes it improper, since the integrand has a vertical asymptote at $x = 1$.

$$\begin{aligned}
\int_1^2 \frac{1}{\sqrt{x-1}}dx &= \lim_{b\to 1^+} \int_b^2 \frac{1}{\sqrt{x-1}}dx \\
(u = x-1) &= \lim_{b\to 1^+} \int_{b-1}^1 \frac{1}{\sqrt{u}}du \\
(c = b-1) &= \lim_{c\to 0^+} \int_c^1 u^{-1/2}du \\
&= \lim_{c\to 0^+} 2u^{1/2}\Big|_c^1 \\
&= \lim_{c\to 0^+} 2\sqrt{1} - 2\sqrt{c} \\
&= 2.
\end{aligned}$$

Therefore, $\displaystyle\int_1^2 \frac{1}{\sqrt{x-1}}dx$ **converges to** 2.

(b) (12 points) Does the improper integral $\displaystyle\int_2^\infty \frac{1}{\sqrt{x-1}}dx$ converge? If yes, to what?

SOLUTION:
The upper bound of this integral makes it improper, since it requires evaluation at infinity. Similar to above, we have

$$\begin{aligned}
\int_2^\infty \frac{1}{\sqrt{x-1}}dx &= \lim_{b\to\infty} \int_1^b \frac{1}{\sqrt{x-1}}dx \\
(\text{as above}) &= \lim_{c\to\infty} 2u^{1/2}\Big|_2^\infty \\
&= \lim_{c\to 0} 2\sqrt{c} - 2\sqrt{1} \\
&= +\infty
\end{aligned}$$

Because the limit is not a number, $\displaystyle\int_2^\infty \frac{1}{\sqrt{x-1}}dx$ **diverges**.

4. Evaluate the following integrals.

(a) (12 points) $\displaystyle\int_1^4 \frac{1}{\sqrt{y}(2y-\sqrt{y})}\,dy$

SOLUTION:

Let $u=\sqrt{y} \Leftrightarrow du = \frac{dy}{2\sqrt{y}}$

$$\int_1^4 \frac{1}{\sqrt{y}(2y-\sqrt{y})}\,dy = \int_1^2 \frac{2}{2u^2-u}\,du = \int_1^2 \frac{A}{u}+\frac{B}{2u-1}\,du$$

Solve for constants:

$A(2u-1)+Bu=2$

Choose: $x=0$

$A(-1)=2 \Leftrightarrow A=-2$

Choose: $x=\frac{1}{2}$

$B(\frac{1}{2})=2 \Leftrightarrow B=4$

We have:

$$\begin{aligned}\int_1^2 \frac{-2}{u}+\frac{4}{2u-1}\,du &= -2\ln|u| + 2\ln|2u-1|\Big|_1^2 \\ &= (-2\ln(2)+2\ln(3)) - (0+2(0)) \\ &= 2\ln(3)-2\ln(2) \\ &= \ln\left(\frac{9}{4}\right)\end{aligned}$$

(b) (12 points) $\displaystyle\int \tan^3 t\,dt$

SOLUTION:

$$\begin{aligned}\int \tan^3 t\,dt = \int (\tan^2 t)(\tan t\,dt) &= \int (\sec^2 t - 1)\tan t\,dt \\ &= \int \sec^2 t\tan t\,dt - \int \tan t\,dt\end{aligned}$$

Let $u=\tan t \Rightarrow du = \sec^2 t\,dt$ Let $v=\cos t \Rightarrow dv = -\sin t\,dt$

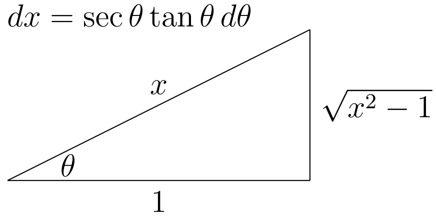
$$\begin{aligned}&= \int u\,du + \int \frac{1}{v}\,dv \\ &= \frac{1}{2}u^2 + \ln|v| + C \\ &= \frac{1}{2}\tan^2 t + \ln|\cos t| + C\end{aligned}$$

(c) (12 points) $\displaystyle\int \frac{dx}{x(x^2-1)^{\frac{3}{2}}} \quad (x > 1)$

SOLUTION:

$x = \sec\theta \quad \Leftrightarrow \quad \theta = \operatorname{arcsec} x$ OR $\theta = \arccos(1/x)$

$dx = \sec\theta\tan\theta\, d\theta$



$$\int \frac{dx}{x(x^2-1)^{\frac{3}{2}}} = \int \frac{\sec\theta\tan\theta}{\sec\theta(\sec^2\theta-1)^{\frac{3}{2}}} d\theta = \int \frac{\tan\theta}{(\tan^2\theta)^{\frac{3}{2}}} d\theta$$

$$= \int \cot^2\theta\, d\theta = \int \frac{1-\sin^2\theta}{\sin^2\theta}\, d\theta$$

$$= \int \csc^2\theta\, d\theta - \int d\theta$$

$$= -\cot\theta - \theta + C$$

$$= \frac{-1}{x^2-1} - \operatorname{arcsec} x + C$$

$$\text{OR} = \frac{-1}{x^2-1} - \arccos(1/x) + C$$