1. [APPM 1360 Exam (36 pts)] Evaluate the following integrals.

(a) \[ \int \sqrt{x} \ln x \, dx \]
(b) \[ \int_0^\infty \frac{x^2}{(4-x^2)^{3/2}} \, dx \]
(c) \[ \int_2^\infty \frac{dv}{v^2+6v-7} \]

**SOLUTION:**

(a) Use integration by parts with \( dv = \sqrt{x} \, dx \), \( v = \frac{2}{3}x^{3/2} \) and \( u = \ln x \), \( du = \frac{1}{x} \, dx \). Then

\[
\int \sqrt{x} \ln x \, dx = \frac{2}{3}x^{3/2} \ln x - \int \frac{2}{3} \frac{x^{3/2}}{x} \, dx = \frac{2}{3}x^{3/2} \ln x - \frac{4}{9}x^{3/2} + C = \frac{2}{3}x^{3/2} \left( \ln x - \frac{2}{3} \right) + C
\]

(b) Use the trig substitution \( x = 2 \sin \theta \). Then \( du = 2 \cos \theta \, d\theta \). For the bounds, \( x = 0 \implies \theta = 0 \) and \( x = \sqrt{2} \implies \theta = \pi/4 \). Then

\[
\int_0^{\pi/4} \tan^2 \theta \, d\theta = \int_0^{\pi/4} (\sec^2 \theta - 1) \, d\theta = (\tan \theta - \theta) \bigg|_0^{\pi/4} = 1 - \pi/4
\]

(c) This is an improper integral and to integrate we’ll need to use partial fractions.

\[
\frac{1}{v^2+6v-7} = \frac{1}{(v+7)(v-1)} = \frac{A}{v+7} + \frac{B}{v-1} \implies 1 = A(v-1) + B(v+7)
\]

\[
v = 1: 1 = B(8) \implies B = \frac{1}{8}
\]

\[
v = -7: 1 = A(-8) \implies A = -\frac{1}{8}
\]

\[
\int_2^{\infty} \frac{dv}{v^2+6v-7} = \frac{1}{8} \lim_{t \to \infty} \int_2^t \left( \frac{1}{v-1} - \frac{1}{v+7} \right) \, dv = \frac{1}{8} \lim_{t \to \infty} \left( \ln |v-1| - \ln |v+7| \right) \bigg|_2^t
\]

\[
= \frac{1}{8} \lim_{t \to \infty} \ln \left| \frac{v-1}{v+7} \right| \bigg|_2^t = \frac{1}{8} \lim_{t \to \infty} \left( \ln \left| \frac{t-1}{t+7} \right| - \ln \left| \frac{2}{2+7} \right| \right)
\]

\[
= \frac{1}{8} \lim_{t \to \infty} \left( \ln \left| 1 - \frac{1}{t+7/t} \right| - \ln \frac{1}{9} \right) = \frac{1}{8} \ln \left( 1 + \ln 9 \right) = \frac{\ln 9}{8}
\]

2. [APPM 1360 Exam (20 pts)] The following problems are not related.

(a) (12 pts) Determine whether \( \int_0^2 \frac{1}{x^4 + \sqrt{x}} \, dx \) converges or diverges. Justify your answer completely.

(b) (8 pts) Write down the form of the partial fraction decomposition of the following function. Do not solve for the coefficients.

\[ R(x) = \frac{x^4 + 1}{x^3(x+1)(x^2+3)^2} \]

**SOLUTION:**

(a) Since we cannot integrate this directly, we use the Comparison Theorem. We have

\[ 0 \leq \sqrt{x} \leq x^4 + \sqrt{x} \implies 0 \leq \frac{1}{x^4 + \sqrt{x}} \leq \frac{1}{\sqrt{x}} \]

Now \( \int_0^2 \frac{1}{\sqrt{x}} \, dx \) converges since this is a \( p \)-integral with \( p = 1/2 < 1 \). Thus, \( \int_0^2 \frac{1}{x^4 + \sqrt{x}} \, dx \) converges by the Comparison Theorem.
3. Your calculator is broken and you have a need to compute \( \ln 4 \). With your newfound knowledge of approximate integration, you decide to do this using the definition of the natural logarithm, that is, you will use the fact that \( \ln 4 = \int_{1}^{4} \frac{1}{x} \, dx \).

(a) Use the Trapezoidal Rule to estimate the value of the integral with 3 subintervals.

(b) Suppose you have to know \( \ln 4 \) with an error no greater than \( 10^{-6} \). Using the above integral, how many subintervals would be necessary, using the Midpoint Rule, to estimate \( \ln 4 \) with this accuracy?

**SOLUTION:**

(a) We have \( \Delta x = \frac{b - a}{n} = \frac{4 - 1}{3} = 1 \). This then gives

\[
T_3 = \frac{1}{2} \left[ \frac{1}{1} + 2 \left( \frac{1}{2} \right) + 2 \left( \frac{1}{3} \right) + \frac{1}{4} \right] = \frac{35}{24}
\]

(b) We need to find an upper bound on the absolute value of the second derivative of \( f(x) = \frac{1}{x} \) on the interval \( 1 \leq x \leq 4 \). We have

\[
f(x) = x^{-1} \implies f'(x) = -1x^{-2} \implies f''(x) = 2x^{-3} \implies |f''(x)| = 2x^{-3} = 2/x^3
\]

which is a strictly decreasing function on \([1, 4]\). Therefore, it takes on its maximum value at the left endpoint \( x = 1 \) so that \( K = 2 \), that is, \( |f''(x)| \leq 2 \). Then we need

\[
|E_M| \leq \frac{K(b - a)^3}{24n^2} < 10^{-6}
\]

\[
\frac{2(4 - 1)^3}{24n^2} < 10^{-6}
\]

\[
\frac{9(10^6)}{4} < n^2
\]

\[
1500 < n
\]

4. Consider the region bounded by the curves \( y = x^2 \) and \( y = 2 - x \).

(a) Graph the region, labeling intersection points.

(b) Set up, but do not evaluate, the integral(s) to find the area of the region if integrating with respect to \( y \).

(c) Set up, but do not evaluate, the integral(s) to find the area of the region if integrating with respect to \( x \).

(d) Find the actual area by evaluating one of the integrals from part (a) or (b). Be sure to clearly indicate which part you are using.

**SOLUTION:**

(a) Sketch of area.

(b) \[
\int_{0}^{1} [\sqrt{y} - (-\sqrt{y})] \, dy + \int_{1}^{4} [(2 - y) - (-\sqrt{y})] \, dy
\]
(c) 

$$\int_{-2}^{1} [(2 - x) - x^2] \, dx$$

(d) Integration with respect to $x$:

$$\int_{-2}^{1} [(2 - x) - x^2] \, dx = \left[ 2x - \frac{1}{2} x^2 - \frac{1}{3} x^3 \right]_{-2}^{1} = \left( 2 - \frac{1}{2} - \frac{1}{3} \right) - \left( -4 - 2 + \frac{8}{3} \right) = \frac{9}{2}$$

Integration with respect to $y$:

$$\int_{0}^{1} [\sqrt{y} - (-\sqrt{y})] \, dy + \int_{1}^{4} [(2 - y) - (-\sqrt{y})] \, dy = \int_{0}^{1} 2y^{1/2} \, dy + \int_{1}^{4} (2 - y + y^{1/2}) \, dy$$

$$\left. \frac{4}{3} y^{3/2} \right|_{0}^{1} + \left. \left( 2y - \frac{1}{2} y^2 + \frac{2}{3} y^{3/2} \right) \right|_{1}^{4} = \frac{4}{3} + \left[ 8 - 8 + \frac{16}{3} - \left( 2 - \frac{1}{2} + \frac{2}{3} \right) \right] = \frac{9}{2}$$