1. (25 pts) The following parts are not related.

(a) Evaluate the integral \( \int \frac{dx}{x^2 \sqrt{1 + x^2}} \)

(b) Find an error bound for the Trapezoid Rule approximation to \( \int_{-3}^{-1} \frac{1}{2x^2} \, dx \) when \( n = 2 \).

Solution:

(a) (13 pts) Use trig-sub.

\[ \tan \theta = x \implies \sec^2 \theta \, d\theta = dx \]

\[ \frac{1}{\sqrt{x^2 + 1}} = \cos \theta \implies \sqrt{x^2 + 1} = \sec \theta \]

The integral is

\[ \int \frac{dx}{x^2 \sqrt{1 + x^2}} = \int \frac{\sec^2 \theta \, d\theta}{\tan^2 \theta \cdot \sec \theta} = \int \frac{\cos \theta \, d\theta}{\sin \theta} = \int \frac{1}{u^2} \, du = -\frac{1}{\sin \theta} + C = -\frac{\sqrt{1 + x^2}}{x} + C \]

(b) (12 pts) The error bound is given by

\[ |E_T| \leq \frac{K(b-a)^3}{12n^2}, \quad K = \max_{x \in [-3,-1]} |f^{(2)}(x)| \]

Now

\[ \frac{d^2}{dx^2} \left( \frac{1}{2x^2} \right) = \frac{3}{x^4} \]

and

\[ \frac{d}{dx} \frac{3}{x^4} = -\frac{12}{x^5} \geq 0 \quad \text{if} \quad -3 \leq x \leq -1 \]

so \( f^{(2)}(x) \) is an increasing function on \([-3,-1]\). To maximize this function evaluate at the right endpoint of the interval, \( x = -1 \):

\[ K \leq \left. \frac{d^2}{dx^2} \left( \frac{1}{2x^2} \right) \right|_{x=-1} = \left. \frac{3}{x^4} \right|_{x=-1} = 3 \]

The error bound is

\[ |E_T| \leq \frac{3 \cdot 2^3}{12 \cdot (2)^2} = \frac{1}{2} \]
2. (25 pts) The following parts are not related.

(a) Find the general solution to the differential equation \( \frac{dy}{dt} = -3yt^2 \).

(b) Consider a thin metal plate of uniform density defined by the region bounded by \( y = \sqrt{\ln x} \) and \( y = \sqrt{2x} \) for \( 1 \leq x \leq e \). If the moment about the \( x \)-axis is \( M_x = 3e^2 - 6 \) find the density of the plate.

Solution:

(a) (10 pts) Use Separation of Variables

\[
\frac{dy}{dt} = -3yt^2 \iff \frac{dy}{y} = -3t^2 dt
\]

\[
\int \frac{dy}{y} = \int -3t^2 dt
\]

\[
\ln |y| = -t^3 + c
\]

\[
y(t) = Ce^{-t^3}
\]

(b) (15 pts) The region in question is

\[
M_x = \rho \frac{2}{2} \int_a^b (f(x)^2 - g(x)^2) dx
\]

\[
= \rho \frac{2}{2} \int_1^e \left( (\sqrt{2x})^2 - (\sqrt{\ln x})^2 \right) dx
\]

\[
= \rho \frac{2}{2} \int_1^e (2x - \ln x) dx \quad \text{(IBP: } u = \ln x, \ dv = dx) \]

\[
= \rho \frac{2}{2} \left[ x^2 - x \ln x + x \right]_1^e
\]

\[
= \rho \frac{2}{2} (e^2 - 2)
\]

\[
M_x = 3e^2 - 6 \implies \frac{\rho}{2} (e^2 - 2) \equiv 3e^2 - 6
\]

\[
\implies \rho = 6
\]
3. (50 pts) The following parts are not related. Cite any theorems that you use.

(a) Consider the function \( f(x) = 2^x \).

i. Find the Taylor series for \( f(x) \) centered at \( \alpha = 1 \). (Hint: \( \frac{d}{dx} 2^x = \ln 2 \cdot 2^x \)).

ii. What can you conclude about the series’ radius of convergence if \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 0 \)?

iii. Find the exact error of the \( T_3 \) approximation to \( f(1) \).

iv. Find an error bound for the \( T_3 \) approximation to \( 2^{1/2} \).

(b) Answer the following with “true” or “false.” If the statement is true justify by citing a theorem. If the statement is false give an example demonstrating it does not always hold.

i. If \( b_n \geq 0 \) for every \( n \) and \( \lim_{n \to \infty} b_n = 0 \) then \( \{(-1)^n \cdot b_n\} \) must converge to 0.

ii. If \( \sum a_n \) and \( \sum b_n \) are divergent then \( \sum (a_n \cdot b_n) \) must also diverge.

iii. If \( \{a_n\} \) converges then \( \sum a_n \) converges.

iv. If \( \sum a_n = 3 \) then \( \lim_{n \to \infty} a_n = 0 \).

Solution:

(a) (30 pts)

i. Find the Taylor coefficients:

\[
\begin{align*}
 f^{(0)}(x) &= 2^x \quad \implies f^{(0)}(1) = 2 \\
 f^{(1)}(x) &= \ln 2 \cdot 2^x \quad \implies f^{(1)}(1) = 2 \ln 2 \\
 f^{(2)}(x) &= (\ln 2)^2 \cdot 2^x \quad \implies f^{(2)}(1) = 2(\ln 2)^2 \\
 &\vdots \\
 f^{(n)}(x) &= (\ln 2)^n \cdot 2^x \quad \implies f^{(n)}(1) = 2(\ln 2)^n
\end{align*}
\]

The Taylor series is

\[
f(x) = \sum_{n=0}^\infty \frac{f^{(n)}(1)}{n!} (x-1)^n = \sum_{n=0}^\infty \frac{2(\ln 2)^n}{n!} (x-1)^n
\]

ii. This is an application of the Ratio Test to the power series. We conclude the series converges for every choice of \( x \implies R = \infty \).

iii. Recall that a Taylor series gives the exact value of the function \( f(x) \) when evaluated at the series’ center value. Since \( x = 1 \) is the center of the series in (a), the error is \( R_3 = 0 \). (ie \( f(1) = 2 \), \( T_3(1) = 2 \) and \( |f(1) - T_3(1)| = 0 \)).

iv. From the Taylor Remainder Theorem

\[
|R_3| \leq \frac{f^{(4)}(z)}{4!} (x-1)^4
\]

Note that \( f^{(4)}(z) = (\ln 2)^4 \cdot 2^z \) is an increasing function on the interval \( z \in (\alpha - x, \alpha) = \left( \frac{1}{2}, 1 \right) \) and is maximized at the right endpoint \( z = 1 \).

\[
|R_3| \leq \frac{f^{(4)}(1)}{4!} \left( \frac{1}{2} \right)^4 = \frac{2(\ln 2)^4}{4!} \cdot \frac{1}{2^4} = \frac{(\ln 2)^4}{192} \approx 0.0012
\]
(b) (20 pts)

i. True. Theorem 8.1.6: If \( \lim_{n \to \infty} |a_n| = 0 \) then \( \lim_{n \to \infty} a_n = 0 \) also. Since the limit exists and is finite, the sequence converges to the limiting value, 0.

ii. False. (Numerous examples possible) \( \sum_{n=1}^{\infty} (-1)^n \) diverges by the Test for Divergence and \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges by the P-Series Test. However, \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \) converges by AST.

iii. False. Let \( a_n = 1 \) for every \( n \). Then the sequence converges since \( \lim_{n \to \infty} a_n = 1 \) but \( \sum a_n \) diverges by the Test for Divergence.

iv. True. Theorem 8.2.6: If \( \sum a_n \) converges then \( \lim_{n \to \infty} a_n = 0 \).

4. (50 pts) The following parts are not related.

(a) Consider the curve defined by the parametric equation

\[
C = \begin{cases} 
  x(t) = \sin t \\
  y(t) = \cos^2 t 
\end{cases} \quad 0 \leq t \leq 2\pi
\]

i. Find the Cartesian representation of \( C \) by eliminating the parameter \( t \).

ii. Sketch the parametric curve \( C \) for \( 0 \leq t \leq 2\pi \).

iii. Use a plot of \( x(t) \) against \( t \) to discuss the directions of motion that a particle would travel on \( C \) as the parameter \( t \) increases.

iv. Use the parametric formula to set-up the integral to compute the area bounded by \( C \) and the \( x \)-axis.

v. Set-up an integral to compute the volume of the solid generated by rotating \( C \) about the \( y \)-axis.

(b) Consider the function \( r = 1 + 2 \sin \theta \). Sketch the curve on \( 0 \leq \theta \leq 2\pi \).

**Solution:**

(a) (36 pts)

i. To eliminate the parameter, notice that \( y(t) = \cos^2 t = 1 - \sin^2 t \). Since \( x(t) = \sin t \) then

\[
y = 1 - x^2, \quad -1 \leq x \leq 1
\]

(ii-iii) From the Cartesian equation we know that the curve will be a parabola with \( x \)-intercepts \( x = \pm 1 \) and \( y \)-intercept \( y = 1 \). To see how a particle moves on this curve, examine a plot of \( x(t) \) against \( t \).

- If \( t \in (0, \frac{\pi}{2}) \) the particle is moving down the right branch of the parabola from \( (0,1) \to (1,0) \).
- If \( t \in (\frac{\pi}{2}, \pi) \) the particle returns to its initial position by moving up the right branch of the parabola from \( (1,0) \to (0,1) \).
- If \( t \in (\pi, \frac{3\pi}{2}) \) the particle is moving down the left branch of the parabola from \( (0,1) \to (-1,0) \).
- If \( t \in (\frac{3\pi}{2}, 2\pi) \) the particle returns to its initial position by moving up the left branch of the parabola from \( (-1,0) \to (0,1) \).
iv. Find the area under the curve in the first quadrant \((0 \leq t \leq \frac{\pi}{2})\) and then double it to find the total area (symmetry). It is easiest to use a vertical test strip.

\[
A = 2 \int_{0}^{1} y \, dx = 2 \int_{x=0}^{x=1} y(t) \frac{dx}{dt} \, dt = 2 \int_{0}^{\pi/2} \left( \cos^2 t \right) \left( \cos t \right) dt = 2 \int_{0}^{\pi/2} \cos^3 t \, dt
\]

v. Revolve the branch of the parabola in the first quadrant about the \(y\)-axis. The orientation of the test strip depends on the method used.

- Using the parametric equations and the Disk Method (requires a horizontal test strip):

\[
V_{\text{Disk}} = \pi \int_{y=0}^{y=1} x(t)^2 \frac{dy}{dt} \, dt = \pi \int_{\pi/2}^{0} \left( \sin t \right)^2 \left( -2 \cos t \sin t \right) dt = 2 \pi \int_{0}^{\pi/2} \sin^3 t \cos t \, dt
\]

- Using the Cartesian equation and the Disk Method (requires a horizontal test strip):

\[
V_{\text{Disk}} = \int_{c}^{d} \pi g(y)^2 \, dy = \pi \int_{0}^{1} \left( \sqrt{1-y} \right)^2 \, dy = \pi \int_{0}^{1} \left( 1 - y \right) \, dy
\]

- Using the Cartesian equation and the Cylindrical Shells method (requires a vertical test strip):

\[
V_{\text{Shells}} = \int_{a}^{b} 2\pi x f(x) \, dx = 2\pi \int_{0}^{1} x \left( 1 - x^2 \right) \, dx
\]

(b) (14 pts) Plot the Cartesian graph on the \(\theta r\)-axis and use this to plot the polar graph.

(Note that \(r = 0\) if \(\theta = \frac{7\pi}{6}, \frac{11\pi}{6}\).)