1. Consider the region \( R \) in the first quadrant bounded by the curves \( y = 2x, y = 2x - 6, \) and \( y = 2. \)

(a) Use the Washer/Disk Method to set-up the integral(s) that computes the volume of the solid obtained by rotating the region \( R \) about the axis \( y = 3. \) (Do not evaluate the integral(s).)

(b) Use the Cylindrical Shells Method to set-up the integral(s) that computes the volume of the solid obtained by rotating the region \( R \) about the axis \( y = 3. \) (Do not evaluate the integral(s).)

**Solution:** (20 pts) The region looks like

(a) (12 pts) **Washer-Disk Method:** \( V = \pi \int_{a}^{b} [R^2 - r^2] \, dx. \)

\[
V = V_1 + V_2 + V_3 \quad \text{where}
\]

\[
V_1 = \pi \int_{0}^{1} [(3 - 0)^2 - (3 - 2x)^2] \, dx = \pi \int_{0}^{1} [3^2 - (3 - 2x)^2] \, dx
\]

\[
V_2 = \pi \int_{1}^{3} [(3 - 0)^2 - (3 - 2)^2] \, dx = \pi \int_{1}^{3} [3^2 - 1^2] \, dx
\]

\[
V_3 = \pi \int_{3}^{4} [(3 - (2x - 6))^2 - (3 - 2)^2] \, dx = \pi \int_{3}^{4} [(9 - 2x)^2 - 1^2] \, dx
\]

(b) (8 pts) **Cylindrical Shells Method:** \( V = 2\pi \int_{c}^{d} r(y)h(y) \, dy. \)

\[
V = 2\pi \int_{0}^{2} (3 - y) \left( \frac{y + 6}{2} - \frac{y}{2} \right) \, dy = 2\pi \int_{0}^{2} (3 - y)(3) \, dy = 6\pi \int_{0}^{2} (3 - y) \, dy
\]
2. (a) Find the exact length of the curve \( y = \frac{x^2}{2} - \frac{1}{4} \ln x \) on \( 1 \leq x \leq 2 \).

(b) Find the area of the surface generated by rotating \( y = 2 \sec(x) \) (\( 0 \leq x \leq \frac{\pi}{3} \)) about the \( x \)-axis.

(Hint: Use the identity \( \tan^2(x) = \sec^2(x) - 1 \) and the fact that \( 4u^4 - 4u^2 + 1 \) is a perfect square.)

Solution:

(a) (10 pts) Arc length is defined as

\[
L = \int_a^b \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx
\]

Note

\[
1 + \left( \frac{dy}{dx} \right)^2 = 1 + \left( x - \frac{1}{4x} \right)^2 = 1 + \left( x^2 - \frac{1}{2} + \frac{1}{16x^2} \right) = x^2 + \frac{1}{2} + \frac{1}{16x^2} = \left( x + \frac{1}{4x} \right)^2
\]

So

\[
L = \int_1^2 \sqrt{x + \frac{1}{4x}} \, dx = \int_1^2 \left( x + \frac{1}{4x} \right) \, dx = \left[ \frac{x^2}{2} + \frac{1}{4} \ln x \right]_{x=2}^{x=1} = \frac{3}{2} + \frac{1}{4} \ln(2)
\]

(b) (15 pts) The surface area of a solid of revolution about the \( x \)-axis is

\[
SA = 2\pi \int_a^b r \, ds = 2\pi \int_a^b f(x) \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx
\]

The radius of the solid of revolution is \( f(x) = 2 \sec(x) \). The arc length of the solid is

\[
ds = \sqrt{1 + \left( 2 \sec(x) \tan(x) \right)^2} \, dx
\]

\[
= \sqrt{1 + 4 \sec^2(x)(\sec^2(x) - 1)} \, dx
\]

\[
= \sqrt{4 \sec^4(x) - 4 \sec^2(x) + 1} \, dx
\]

\[
= 2 \sec^2(x) - 1 \, dx
\]

The surface area is

\[
SA = 2\pi \int_0^{\pi/3} (2 \sec(x))(2 \sec^2(x) - 1) \, dx
\]

\[
= 4\pi \int_0^{\pi/3} (2 \sec^3(x) - \sec(x)) \, dx
\]

\[
= 4\pi \sec(x) \tan(x) \bigg|_{0}^{\pi/3}
\]

\[
= 8\sqrt{3}\pi
\]
3. (a) Consider a thin metal plate defined by \( y = 2e^x \) and \( y = 1 \) on \( 0 \leq x \leq 1 \). Suppose that this plate is of density \( \rho(x) = x \). Find the moment about the \( x \)-axis, \( M_x \).

(b) Consider the region of uniform density \( \rho = 1 \) composed of a quarter circle, a square, and an isosceles triangle. Find the centroid for the region (the centroids of each smaller region are given).

Solution:

(a) (12 pts) The moment about the \( x \)-axis:

\[
M_x = \int_0^1 \frac{x}{2} [(2e^x)^2 - 1^2] \, dx
\]

\[
= \frac{1}{2} \int_0^1 [4xe^{2x} - x] \, dx
\]

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\[
= \frac{1}{4} + \frac{1}{2}e^2
\]

(b) (13 pts) First find the masses of each shape:

\[
\begin{align*}
    m_T &= \rho \cdot \frac{1}{2}bh = \frac{1}{2}(2)(1) = 1 \\
    m_S &= \rho \cdot bh = (1)(1) = 1 \\
    m_C &= \rho \cdot \frac{1}{4}\pi r^2 = \frac{\pi}{4}
\end{align*}
\]

The moments about the \( x \) and \( y \)-axes are the sums of the moments of each disjoint region (the moments of each region are \( M_y = mx \) and \( M_x = my \)):

\[
M_y = m_T \cdot x_T + m_S \cdot x_S + m_C \cdot x_C = (1)(-\frac{1}{3}) + (1)(\frac{1}{2}) + (\frac{\pi}{4})(\frac{4}{3\pi}) = \frac{1}{2}
\]

\[
M_x = m_T \cdot y_T + m_S \cdot y_S + m_C \cdot y_C = (1)(0) + (1)(-\frac{1}{2}) + (\frac{\pi}{4})(\frac{4}{3\pi}) = -\frac{1}{6}
\]

The COM is

\[
(\bar{x}, \bar{y}) = \left( \frac{2}{8 + \pi}, \frac{2}{3(8 + \pi)} \right)
\]
4. (a) Do the following sequences converge or diverge? Fully justify your answers and cite any theorems that you use. If any sequence converges, find its limit.

i. \( \left\{ \frac{n}{\ln n^2} \right\}_{n=2}^{\infty} \)

ii. \( \left\{ \sin(\pi n) \right\}_{n=1}^{\infty} \)

iii. \( \left\{ \frac{(n-1)!}{n^n} \right\}_{n=2}^{\infty} \)

(b) Consider the sequence \( a_n = \arctan(2n), \ n \geq 0 \).

i. Is the sequence \( \left\{ a_n \right\} \) monotonic? Fully justify your answer.

ii. Is the sequence \( \left\{ a_n \right\} \) bounded? Fully justify your answer.

iii. Based on your answers to (i) and (ii), does \( \left\{ a_n \right\} \) converge, diverge, or can it not be determined? Fully justify your response and be sure to cite any theorems you use.

Solution:

(a) (20 pts)

i. (8 pts) Notice \( \lim_{n \to \infty} \frac{n}{\ln n^2} = \frac{\infty}{\infty} \) is an indeterminate form. We need to define a corresponding function before taking a derivative:

\[
\text{Set } f(x) = \frac{x}{\ln x^2} \implies \lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{x}{2 \ln x} \]

\[
\overset{\text{LH}}{= \lim_{x \to \infty}} \frac{1}{2/x} = \lim_{x \to \infty} \frac{x}{2} = \infty
\]

Since \( \lim_{x \to \infty} f(x) = \lim_{n \to \infty} a_n \) then \( \lim_{n \to \infty} a_n = \infty \) also. Thus, \( \left\{ a_n \right\} \) diverges.

ii. (4 pts) Notice \( a_n = \sin(\pi n) = 0 \) for every \( n = 1, 2, 3 \ldots \). The sequence can be equivalently written as \( \left\{ 0 \right\}_{n=1}^{\infty} \). Since \( \lim_{n \to \infty} 0 = 0 \) then we conclude the sequence converges to 0.

iii. (8 pts) Since factorials are not continuous functions, we cannot compute \( \lim_{n \to \infty} a_n \) directly. We can find an upper bound for \( a_n \):

\[
\frac{(n-1)!}{n^n} = \frac{(n-1)(n-2)(n-3) \cdots 3 \cdot 2 \cdot 1}{n^n} = \frac{1}{n} \frac{(n-1)}{n} \frac{(n-2)}{n} \frac{(n-3)}{n} \cdots \frac{(3)}{n} \frac{(2)}{n} \frac{(1)}{n}
\]

\[
\leq \frac{1}{n} \frac{(n)}{n} \frac{(n)}{n} \cdots \frac{(n)}{n} \frac{(n)}{n} \frac{(n)}{n} = \frac{1}{n}
\]

Use the Squeeze Theorem to compute the limit:

\[
0 \leq \frac{(n-1)!}{n^n} \leq \frac{1}{n}
\]

\[
\implies \lim_{n \to \infty} 0 \leq \lim_{n \to \infty} \frac{(n-1)!}{n^n} \leq \lim_{n \to \infty} \frac{1}{n}
\]

\[
\implies 0 \leq \lim_{n \to \infty} \frac{(n-1)!}{n^n} \leq 0
\]

\[
\implies \lim_{n \to \infty} a_n = 0
\]

The sequence converges to 0.
(b) (10 pts)

i. (3 pts) Set $f(x) = \arctan(2x) \implies f'(x) = \frac{2}{1 + (2x)^2} > 0$ for every $x \geq 0$. This indicates $a_{n+1} > a_n$ for every $n \geq 0$. The sequence is increasing.

ii. (3 pts) If $n = 0$ then $\arctan(2n) = \arctan(0) = 0$ and if $n \to \infty$ then $\arctan(2n) \to \frac{\pi}{2}$. Thus $0 \leq \arctan(2n) \leq \frac{\pi}{2}$. The sequence is bounded.

iii. (4 pts) Since the sequence is increasing (ie is monotonic) and bounded for every $n \in [0, \infty)$ then by the Monotone Sequence Theorem the sequence converges.