1. (30 pts)

(a) Evaluate the following integrals. Be sure to name the integration technique(s) used.

i. \( \int \frac{x^2}{\sqrt{4 - x^2}} \, dx \)  
ii. \( \int \frac{1}{x^4 + x^2} \, dx \)

(b) Suppose that \( f(x) \) is continuous and differentiable on the interval \( 0 \leq x < \infty \) and that \( \lim_{x \to \infty} x f(x) = 0 \). Is the following relationship true or false? Fully justify your answer.
(Hint: use Integration by Parts).

\[ \int_0^\infty f(x) \, dx = -\int_0^\infty x f'(x) \, dx \]

Solution:

(a) (13 pts)

i. Use a trig substitution. First label a triangle: \( y = \sqrt{4 - x^2} \) \( \iff \) \( y^2 = 4 - x^2 \) \( \iff \) \( y^2 + x^2 = 4 \). So the legs are \( x \) and \( \sqrt{4 - x^2} \) and the hypotenuse is 2.

\[ \text{Identities relating } x \text{ and } \theta \ 	ext{are} \ x = 2 \sin(\theta) \ \
\implies dx = 2 \cos(\theta) \, d\theta \ \
\sqrt{4 - x^2} = 2 \sec(\theta) \]

\[ \int \frac{x^2}{\sqrt{4 - x^2}} \, dx = \int \frac{(2 \sin(x))^2}{2 \cos(\theta)} \cdot 2 \cos(\theta) \, d\theta \]

\[ = 4 \int \sin^2(\theta) \, d\theta \]

\[ = 2 \int (1 - \cos(2\theta)) \, d\theta \]

\[ = 2 \left[ \theta - \frac{1}{2} \sin(2\theta) \right] \]

\[ = 2 \left[ \theta - \sin(\theta) \cos(\theta) \right] \]

\[ = 2 \arcsin \left( \frac{x}{2} \right) - \frac{x \sqrt{4 - x^2}}{2} + C \]
ii. (12 pts) Use partial fraction decomposition:

\[
\frac{1}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2+1}
\]

\[1 = Ax(x^2+1) + B(x^2+1) + (Cx + D)x^2\]

\[1 = (A + C)x^3 + (B + D)x^2 + Ax + B\]

\[\Rightarrow 0 = A + C, \quad A = 0 \Rightarrow C = 0\]

\[0 = B + D, \quad B = 1 \Rightarrow D = -1\]

\[\Rightarrow \frac{1}{x^2(x^2+1)} = \frac{1}{x^2} - \frac{1}{x^2+1}\]

\[\Rightarrow \int \frac{1}{x^4 + x^2} \, dx = \int \left( \frac{1}{x^2} - \frac{1}{x^2+1} \right) \, dx\]

\[= -\frac{1}{x} - \arctan(x) + C\]

(b) (5 pts) Using integration by parts where \(u = f(x) \Rightarrow du = f'(x) \, dx\) and \(dv = 1 \, dx \Rightarrow v = x:\)

\[
\int_0^\infty f(x) \, dx = \lim_{t \to \infty} \int_0^t f(x) \, dx
\]

\[= \lim_{t \to \infty} xf'(x) \bigg|_0^t - \lim_{t \to \infty} \int_0^t xf'(x) \, dx\]

\[= \lim_{t \to \infty} tf'(t) - 0 \cdot f'(0) - \lim_{t \to \infty} \int_0^t xf'(x) \, dx\]

\[= -\int_0^\infty xf'(x) \, dx\]

2. (30 pts) Consider the integral \( I = \int_1^5 x \cdot e^{-1/x} \, dx.\)

(a) Set-up but do not evaluate the Trapezoid Rule integral approximation of \( I \) using four subintervals.

(b) Set-up but do not evaluate the Midpoint Rule integral approximation of \( I \) using four subintervals.

(c) Find an error bound for the approximation in (a).

(d) How would the error bound in (c) be affected if eight subintervals had been used in the integral approximation? Explain briefly.

Solution:

(a) (8 pts) Trap Rule:

\[
\int_a^b f(x) \, dx \approx \frac{\Delta x}{2} \left[ f(x_0) + 2f(x_1) + \ldots + 2f(x_{n-1}) + f(x_n) \right], \quad \Delta x = \frac{1}{n} (b - a)
\]
Notice \( n = 4 \) and \( \Delta x = \frac{1}{4}(5 - 1) = 1 \) so the subintervals are \([1, 2], [2, 3], [3, 4], [4, 5]\). The approximation is

\[
\int_1^5 xe^{-x} dx \approx \frac{1}{2} \left[ 1 \cdot e^{-1} + 2(2 \cdot e^{-1/2}) + 2(3 \cdot e^{-1/3}) + 2(4 \cdot e^{-1/4}) + 5 \cdot e^{-1/5} \right] \\
= \frac{1}{2} \left[ e^{-1} + 4e^{-1/2} + 6e^{-1/3} + 8e^{-1/4} + 5e^{-1/5} \right]
\]

(b) (8 pts) The Midpoint Rule:

\[
\int_a^b f(x) dx \approx \Delta x \left[ f(\overline{x}_1) + f(\overline{x}_2) + \ldots + f(\overline{x}_{n-1}) + f(\overline{x}_n) \right], \quad \Delta x = \frac{1}{n}(b - a), \quad \overline{x}_i = \frac{1}{2}(x_{i-1} + x_i)
\]

Notice \( n = 4 \) and \( \Delta x = \frac{1}{4}(5 - 1) = 1 \) so the subintervals are \([1, 2], [2, 3], [3, 4], [4, 5]\) and the midpoints of these subintervals are \([\frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}]\). The approximation is

\[
\int_1^5 xe^{-x} dx \approx \frac{1}{2} \left[ \frac{3}{2} e^{-2/3} + \frac{5}{2} e^{-2/5} + \frac{7}{2} e^{-2/7} + \frac{9}{2} e^{-2/9} \right]
\]

(c) (10 pts) First note that \( f''(x) = xe^{-x} = \frac{e^{-x}}{x^3} \) which monotonically decreases over the interval \([1, 5]\). The second derivative is maximized at the left endpoint \( x = 1 \Rightarrow K \leq e^{-1} \). Then

\[
|E_T| \leq \frac{K(b - a)^3}{12n^2} = \frac{e^{-1}(5 - 1)^3}{12 \cdot 4^2} = \frac{e^{-1}}{3}
\]

(d) (4 pts) If eight subintervals were used in the approximation, the error bound would decrease.

3. (25 pts) Explain why the following integrals are improper, then determine if they converge or diverge. If the integral converges, find its value. Be sure to name any theorems that you use.

(a) \( \int_8^\infty \frac{2x}{x^{5/3} - 3} dx \) 

(b) \( \int_0^e x \ln x \, dx \)

Solution:

(a) (12 pts) The integral is improper because one of the bounds is infinite. Notice

\[
x^{5/3} - 3 \leq x^{5/3} \Rightarrow \frac{1}{x^{5/3} - 3} \geq \frac{1}{x^{5/3}} \Rightarrow \frac{2x}{x^{5/3} - 3} \geq \frac{2x}{x^{2/3}} = \frac{2}{x^{2/3}}
\]

\[\Rightarrow \int_8^\infty \frac{2x}{x^{5/3} - 3} dx \geq \int_8^\infty \frac{2}{x^{2/3}} dx\]

We know that \( \int_8^\infty \frac{2}{x^{2/3}} dx \) is a \( p \)-integral on \([8, \infty)\) with \( p = \frac{2}{3} < 1 \). Thus, the test integral diverges. By the Direct Comparison Theorem, the given integral diverges also.
(b) (13 pts) The integral is improper because there is an infinite discontinuity at \(x = 0\).

Since \(\ln x < 0\) for \(0 < x < 1\), the Comparison Theorem does not apply. Evaluating the integral directly

\[
\int_0^e x \ln x \, dx = \lim_{t \to 0^+} \int_t^e x \ln x \, dx
\]

Using Integration by Parts with \(u = \ln x \implies du = \frac{dx}{x}\) and \(dv = x \implies v = \frac{1}{2}x^2\)

\[
\lim_{t \to 0^+} \int_t^e x \ln x \, dx = \frac{1}{2} \lim_{t \to 0^+} x^2 \ln x \bigg|_t^e - \frac{1}{2} \int_0^e x \, dx
\]

\[
= \frac{1}{2} e^2 - \frac{1}{2} \lim_{t \to 0^+} t^2 \ln t - \frac{1}{4} e^2
\]

\[
= \frac{e^2}{4} - \lim_{t \to 0^+} \frac{\ln t}{t^{-2}}
\]

\[
= \frac{e^2}{4} \lim_{t \to 0^+} \frac{t^{-1}}{-2t^{-3}}
\]

\[
= \frac{e^2}{4} < \infty
\]

The integral converges to \(\frac{e^2}{4}\).

4. (15 pts) Consider the region \(\mathcal{R}\) in the first quadrant bounded by \(y = x^2, y = -x^2 + 8\), and the \(y\)-axis.

(a) Sketch the region \(\mathcal{R}\). Be sure to clearly label the curves and any intersection points.

(b) Set up but do not evaluate the integral(s) in terms of \(x\) that computes the area of \(\mathcal{R}\).

You do not need to simplify. (It may help to draw a test strip.)

(c) Set up but do not evaluate the integral(s) in terms of \(y\) that computes the area of \(\mathcal{R}\).

You do not need to simplify. (It may help to draw a test strip.)

Solution:

(a) (3 pts) A sketch of the region is

The intersection points occur when

\[
x^2 = 8 - x^2 \implies 2x^2 = 8 \implies x = 2 \text{ so } y = 4
\]

Also \(x = 0 \implies y = 0, y = 8\)

The intersection points are \(\{(0,0), (0,8), (2,4)\}\)
(b) (6 pts) An integral in terms of $x$ will utilize vertical test strips. The upper bound ("top") is $y = 8 - x^2$ and the lower bound ("bottom") is $y = x^2$. The bounds of integration are $x \in [0, 2]$.

\[
\text{Area} = \int_0^2 \left( (8 - x^2) - x^2 \right) dx
\]

(c) (6 pts) An integral in terms of $y$ will utilize horizontal test strips. There is a change in symmetry at $y = 4$.

- For the interval $y \in [0, 4]$ the upper bound ("right") is $y = x^2 \iff x = \sqrt{y}$ and the lower bound ("left") is $x = 0$.
- For the interval $y \in [4, 8]$ the upper bound ("right") is $y = 8 - x^2 \iff x = \sqrt{8 - y}$ and the lower bound ("left") is $x = 0$.

\[
\text{Area} = \int_0^4 (\sqrt{y} - 0) \, dy + \int_4^8 (\sqrt{8 - y} - 0) \, dy
\]
\[
= \int_0^4 \sqrt{y} \, dy + \int_4^8 \sqrt{8 - y} \, dy
\]