1. (16 points) Determine whether each of the following series is absolutely convergent, conditionally convergent, or divergent. For this problem, and all subsequent problems, explain your work and name any test or theorem that you use.

   (a) $\displaystyle\sum_{n=2}^{\infty}\left(-\frac{1}{2}\right)^n n^3$

   (b) $\displaystyle\sum_{n=1}^{\infty}\frac{n+2}{\sqrt{n^3+5}}$

   Solution:

   (a) Apply the Ratio Test with $a_n = \left(-\frac{1}{2}\right)^n n^3$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\left(-\frac{1}{2}\right)^{n+1}(n+1)^3}{\left(-\frac{1}{2}\right)^n n^3}\right| = \lim_{n\to\infty}\left|\frac{1}{2}\left(\frac{n+1}{n}\right)^3\right| = \lim_{n\to\infty}\left|\frac{1}{2}\left(1+\frac{1}{n}\right)^3\right| = \frac{1}{2} < 1$$

   Thus the series is $\boxed{\text{absolutely convergent}}$.

   (b) Apply the Limit Comparison Test and compare to the divergent p-series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ $\left(p=\frac{1}{2}\right)$.

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{n+2}{\sqrt{n^3+5}}}{\frac{1}{\sqrt{n}}}\lim_{n\to\infty}\frac{n+2}{\sqrt{n^3+5}}\cdot\frac{\sqrt{n}}{1} = \lim_{n\to\infty}\frac{n^{3/2}+2\sqrt{n}}{\sqrt{n^3+5}}\cdot\frac{\frac{1}{n^{3/2}}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty}\frac{1+\frac{2}{n}}{\sqrt{1+\frac{5}{n^3}}} = 1 > 0$$

   Therefore the given series also is $\boxed{\text{divergent}}$.

2. (12 points) Use the Maclaurin series for $\ln(1+x)$ and $\ln(1-x)$ to find the Maclaurin series for $\ln\left(\frac{1+x}{1-x}\right)$. Write your answer using sigma notation and include the radius of convergence.
   (*Hint:* Write out the first few terms of the $\ln(1+x)$ and $\ln(1-x)$ series.)

   Solution:

$$\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$\ln(1-x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(-x)^n}{n} = \sum_{n=1}^{\infty}-\frac{x^n}{n} = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$$

   Note: $(-1)^{n-1}(-1)^n = (-1)^{2n-1} = -1$. Then

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$$

$$= \left(x - \cancel{\frac{x^2}{2}} + \frac{x^3}{3} - \cancel{\frac{x^4}{4}} + \cdots\right) - \left(-x - \cancel{\frac{x^2}{2}} - \frac{x^3}{3} - \cancel{\frac{x^4}{4}} - \cdots\right)$$

$$= \left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right) + \left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right)$$

$$= 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right) = \boxed{\sum_{n=0}^{\infty} \frac{2x^{2n+1}}{2n+1}}.$$

Because both $\ln(1+x)$ and $\ln(1-x)$ series have a radius of $R = 1$, their difference also has a radius of $\boxed{R = 1}$.

3. (18 points)

   (a) Find a series representation for $\int_0^1 e^{-x^3}\, dx$.

   (b) Use the Alternating Series Estimation Theorem to approximate the value of the definite integral from part (a) with an error less than $1/20$. Fully simplify your answer. (You may assume that the hypotheses of the Alternating Series Estimation Theorem are satisfied.)

Solution:

(a) According to the list of frequently-used Maclaurin series on the cover page of the exam, the Maclaurin series for e^x is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}\,, \qquad R = \infty$$

Therefore,

$$e^{-x^3} = \sum_{n=0}^{\infty} \frac{\left(-x^3\right)^n}{n!}\,, \qquad R = \infty$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n}}{n!}\,, \qquad R = \infty$$

Term-by-term integration leads to

$$\int_0^1 e^{-x^3}\, dx = \int_0^1 \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n}}{n!}\, dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} \int_0^1 x^{3n}\, dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} \left[\frac{x^{3n+1}}{3n+1} \Bigg|_0^1 \right]$$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{1}{n!(3n+1)}}$$

(b) The result from part (a) indicates that

$$\begin{aligned}\int_0^1 e^{-x^3}\,dx &= \sum_{n=0}^{\infty}(-1)^n \frac{1}{n!(3n+1)} \\ &= \frac{1}{0!(1)} - \frac{1}{1!(4)} + \frac{1}{2!(7)} - \frac{1}{3!(10)} + \cdots \\ &= 1 - \frac{1}{4} + \frac{1}{14} - \frac{1}{60} + \cdots\end{aligned}$$

This is an alternating series with $b_n = \dfrac{1}{n!(3n+1)}$.

The problem statement ensures that the hypotheses of the Alternating Series Estimation Theorem are satisfied, so that the following result holds:

$$|\text{error}| = |s - s_n| \le b_{n+1}$$

where s is the infinite sum and s_n is the nth partial sum. Since $b_3 = 1/60 < 1/20$, then using s_2 to estimate the value of s produces an acceptable error. Therefore,

$$\int_0^1 e^{-x^3}\,dx \approx s_2 = 1 - \frac{1}{4} + \frac{1}{14} = \frac{28-7+2}{28} = \boxed{\frac{23}{28}}.$$

4. (28 points) Define a function $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} x^n$.

(a) Determine the values of x for which the series is absolutely convergent.

(b) Find a Taylor series for $f'(x)$.

(c) Find a closed form (non-series) expression for $xf'(-x)$.

Solution:

(a) Apply the Ratio Test.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{1}{(n+1)^2}x^{n+1}}{\frac{1}{n^2}x^n}\right| = \lim_{n\to\infty}\left|\left(\frac{n}{n+1}\right)^2 x\right| = |x|\left(\lim_{n\to\infty}\frac{n}{n+1}\right) \overset{LH}{=} |x|$$

The series has a radius $R = 1$ and is absolutely convergent for $|x| < 1$. Next consider the endpoints of the interval $(-1, 1)$.

At $x = 1$, the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ is the absolutely convergent p-series ($p = 2$).

At $x = -1$, the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$ also is absolutely convergent based on the previous result.

Therefore the given series is absolutely convergent for x in $\boxed{[-1, 1]}$.

(b)

$$f(x) = \sum_{n=1}^{\infty}\frac{1}{n^2}x^n$$

$$f'(x) = \frac{d}{dx}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}x^n\right) = \sum_{n=1}^{\infty}\frac{1}{n^2}\cdot\frac{d}{dx}(x^n) = \sum_{n=1}^{\infty}\frac{1}{n^2}\left(nx^{n-1}\right) = \boxed{\sum_{n=1}^{\infty}\frac{1}{n}x^{n-1}}$$

(c)

$$xf'(-x) = x\sum_{n=1}^{\infty}\frac{1}{n}(-x)^{n-1} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n = \boxed{\ln(1+x)}$$

5. (26 points) The following problems are not related.

   (a) Find $T_2(x)$, the second order Taylor polynomial, centered at $\pi/4$, for $f(x) = \sin(x)$.

   (b) Write the series in sigma notation and find its sum.

$$\frac{1}{1!\,3} + \frac{1}{2!\,9} + \frac{1}{3!\,27} + \frac{1}{4!\,81} + \cdots$$

Solution:

(a)

$$\begin{aligned} f(x) &= \sin x & f\left(\tfrac{\pi}{4}\right) &= \tfrac{1}{\sqrt{2}} \\ f'(x) &= \cos x & f'\left(\tfrac{\pi}{4}\right) &= \tfrac{1}{\sqrt{2}} \\ f''(x) &= -\sin x & f''\left(\tfrac{\pi}{4}\right) &= -\tfrac{1}{\sqrt{2}} \end{aligned}$$

$$\begin{aligned} T_2(x) &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 \\ &= f(\pi/4) + \frac{f'(\pi/4)}{1!}\left(x-\frac{\pi}{4}\right) + \frac{f''(\pi/4)}{2!}\left(x-\frac{\pi}{4}\right)^2 \\ &= \boxed{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(x-\frac{\pi}{4}\right) - \frac{1}{2\sqrt{2}}\left(x-\frac{\pi}{4}\right)^2} \end{aligned}$$

(b)

$$\frac{1}{1!\,3} + \frac{1}{2!\,9} + \frac{1}{3!\,27} + \frac{1}{4!\,81} + \cdots = \frac{1}{1!\,3^1} + \frac{1}{2!\,3^2} + \frac{1}{3!\,3^3} + \frac{1}{4!\,3^4} + \cdots = \sum_{n=1}^{\infty}\frac{(1/3)^n}{n!}$$

The series sums to $\boxed{e^{1/3}-1}$ because $\sum_{n=0}^{\infty}\frac{x^n}{n!} = e^x$ which implies $\sum_{n=1}^{\infty}\frac{x^n}{n!} = e^x - 1$.

   (c) Consider the parametric equations given by $x = 1 + \cos t$ and $y = t + \pi$ for $-\pi \le t \le \pi$. Eliminate the parameter and sketch the curve. Indicate with an arrow the direction in which the curve is traced as t increases.

Solution:

Eliminating the parameter gives $y = t + \pi \implies t = y - \pi$, so $\boxed{x = 1 + \cos(y-\pi)}$.

Note that eliminating the parameter from the x equation, then substituting into the y equation, leads to the function $y = \cos^{-1}(x-1)+\pi$, which will produce only the top half of the parametric curve. This is because the $\cos^{-1}(x)$ function has a range of $[0,\pi]$, yielding a y-range of $[\pi, 2\pi]$, not $[0, 2\pi]$.

t	x	y
$-\pi$	0	0
$-\frac{2\pi}{3}$	$\frac{1}{2}$	$\frac{\pi}{3}$
$-\frac{\pi}{2}$	1	$\frac{\pi}{2}$
$-\frac{\pi}{3}$	$\frac{3}{2}$	$\frac{2\pi}{3}$
0	2	π

t	x	y
$\frac{\pi}{3}$	$\frac{3}{2}$	$\frac{4\pi}{3}$
$\frac{\pi}{2}$	1	$\frac{3\pi}{2}$
$\frac{2\pi}{3}$	$\frac{1}{2}$	$\frac{5\pi}{3}$
π	0	2π