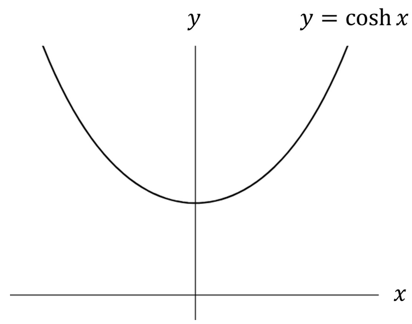
1. (28 pts) The region $\mathcal{R}$ is bounded by the curve $y = \cosh x$ and the x-axis on the interval $0 \leq x \leq \ln 2$.



   (a) Set up but do not evaluate integrals to find the requested quantities.

      i. The volume of a solid with $\mathcal{R}$ as the base and cross-sections perpendicular to the x-axis that are semi-circular regions with diameters in the base.

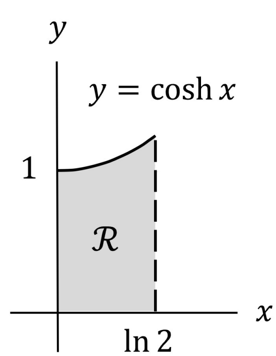
      ii. The volume generated by rotating $\mathcal{R}$ about the line $x = 1$.

   (b) Rotate the curve $y = \cosh x$ on the given interval about the line $x = 1$. Evaluate an integral to find the resulting surface area. You may leave your answer in terms of hyperbolic functions.
   $\big($*Hint:* $\cosh^2 x - \sinh^2 x = 1\big)$

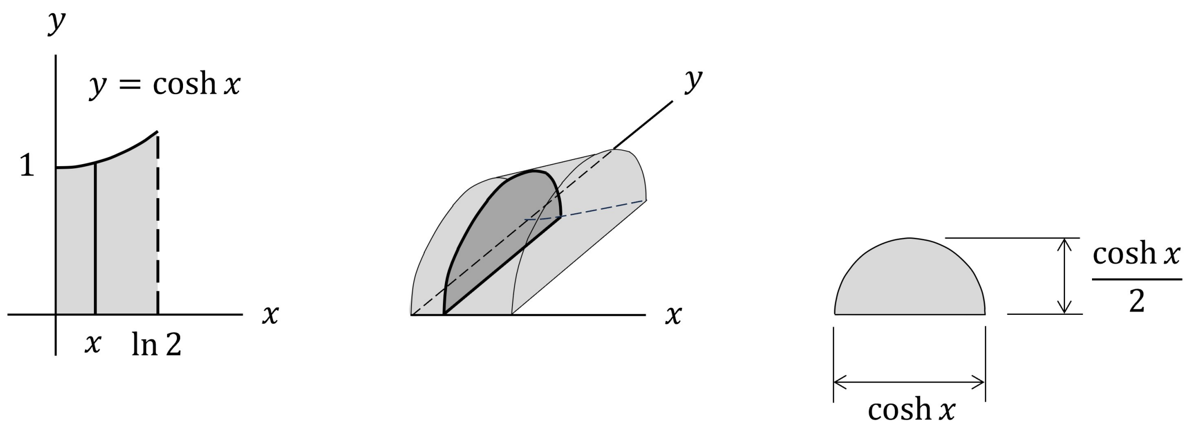
Solution:

(a)

i. The region $\mathcal{R}$ is given by:



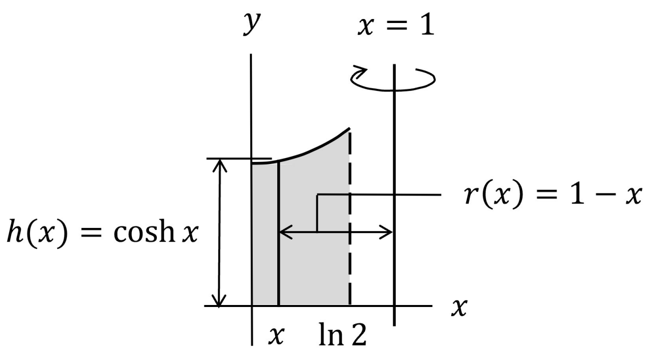
Every x value on the interval $[0, \ln 2]$ is associated with a cross-sectional slice of the solid. The top view, a three-dimensional view, and the front view of a typical slice are depicted below.



Since each cross-section is a semicircle of radius $\frac{\cosh x}{2}$, the cross-sectional area of the slice associated with a particular value of x equals $\frac{\pi}{2}\left(\frac{\cosh x}{2}\right)^2 = \frac{\pi}{8}\cosh^2 x$. Therefore, the volume of the solid can be represented by

$$V = \boxed{\int_0^{\ln 2} \frac{\pi}{8}\cosh^2 x\, dx}$$

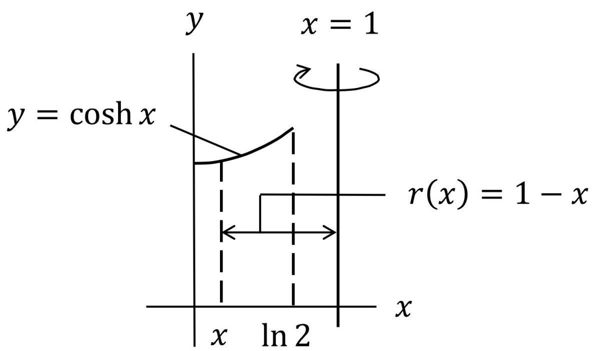
ii. Apply the method of cylindrical shells.



Every x value on the interval $[0, \ln 2]$ is associated with a cylindrical shell. The radius of the shell associated with a particular value of x equals $r(x) = (1 - x)$ and the height of that shell equals $h(x) = \cosh x$.

$$V = \int_0^{\ln 2} 2\pi r(x)h(x)\, dx = \boxed{\int_0^{\ln 2} 2\pi(1 - x)\cosh x\, dx}$$

(b)



$$\frac{dy}{dx} = \frac{d}{dx}[\cosh x] = \sinh x$$

$$\begin{aligned} S &= \int_0^{\ln 2} 2\pi r(x)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \\ &= \int_0^{\ln 2} 2\pi(1-x)\sqrt{1+\sinh^2 x}\,dx \\ &= \int_0^{\ln 2} 2\pi(1-x)\cosh x\,dx \end{aligned}$$

Apply integration by parts with $u = (1-x)$ and $dv = \cosh x\,dx$. Those assignments of u and dv correspond to $du = -dx$ and $v = \sinh x$.

$$\begin{aligned} S &= 2\pi\left[(1-x)\sinh x\Big|_0^{\ln 2} + \int_0^{\ln 2} \sinh x\,dx\right] \\ &= 2\pi\left[(1-x)\sinh x + \cosh x\right]\Big|_0^{\ln 2} \\ &= \boxed{2\pi\left[(1-\ln 2)\sinh(\ln 2) + \cosh(\ln 2) - 1\right]} \end{aligned}$$

Note: This expression can be more fully simplified using $\sinh(\ln 2) = \dfrac{e^{\ln 2} - e^{-\ln 2}}{2} = \dfrac{3}{4}$ and $\cosh(\ln 2) = \dfrac{e^{\ln 2} + e^{-\ln 2}}{2} = \dfrac{5}{4}$. Then, $S = 2\pi[(1-\ln 2)\dfrac{3}{4} + \dfrac{5}{4} - 1] = \boxed{2\pi(1 - (3/4)\ln 2)}$

2. (24 pts) The following two problems are not related.

(a) Find the solution of the equation $\dfrac{dy}{dx} = x\sqrt{1-y^2}$ with initial condition $y(0) = \frac{1}{2}$. Express your answer in the form $y = f(x)$.

(b) A trapezoidal region has vertices at $(0,0), (0,3), (k,3)$, and $(k,1)$, where k is a positive constant. If the x-coordinate of the centroid of the region is $\bar{x} = 7$, find the value of k.

Solution:

(a) This is a separable equation.

$$\frac{dy}{dx} = x\sqrt{1-y^2}$$
$$\int \frac{dy}{\sqrt{1-y^2}} = \int x\,dx$$
$$\sin^{-1} y = \frac{x^2}{2} + C$$

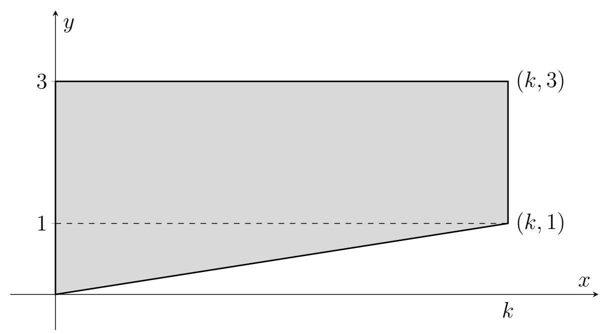
Use the initial value $y(0) = 1/2$ to solve for C.

$$\sin^{-1}\frac{1}{2} = 0 + C = C$$
$$C = \frac{\pi}{6}$$

Therefore the solution is

$$\sin^{-1} y = \frac{x^2}{2} + \frac{\pi}{6}$$
$$y = \sin\left(\frac{x^2}{2} + \frac{\pi}{6}\right).$$

(b) (Note: please compare this to written homework set #6, problem 6.)



The area of the trapezoid is $A = \frac{1}{2}h(b_1 + b_2) = \frac{1}{2}k(3+2) = \frac{5}{2}k$. An equation of the line connecting points $(0,0)$ and $(k,1)$ is $y = \frac{1}{k}x$. It follows that the x-coordinate of the centroid equals

$$\begin{aligned}
\overline{x} = \frac{M_y}{m} &= \frac{1}{A}\int_0^k x\,(f(x) - g(x))\,dx \\
&= \frac{1}{\frac{5}{2}k}\int_0^k x\left(3 - \frac{1}{k}x\right)dx \\
&= \frac{2}{5k}\int_0^k \left(3x - \frac{1}{k}x^2\right)dx \\
&= \frac{2}{5k}\left[\frac{3}{2}x^2 - \frac{x^3}{3k}\right]_0^k \\
&= \frac{2}{5k}\left(\frac{3}{2}k^2 - \frac{k^2}{3}\right) \\
&= \frac{2}{5k}\cdot\frac{7}{6}k^2 = \frac{7}{15}k
\end{aligned}$$

We are given that $\overline{x} = 7$, so $k = 15$.

Alternate Solution:

Divide the trapezoidal region at $y = 1$ to form a rectangle and a right triangle. The rectangle has mass $m_1 = \rho(2k)$ and centroid at $(\overline{x_1}, \overline{y_1}) = (k/2, 2)$. The triangle has mass $m_2 = \rho(k/2)$ and centroid at $(\overline{x_2}, \overline{y_2}) = (k/3, 2/3)$. (The centroid of a triangle is located 2/3 of the way along the median from a vertex to the opposite side.) Using additivity of moments, the centroid of the trapezoidal region has an x-coordinate of

$$\overline{x} = \frac{m_1\overline{x_1} + m_2\overline{x_2}}{m_1 + m_2} = \frac{\rho}{\rho} \cdot \frac{(2k)(k/2) + (k/2)(k/3)}{2k + k/2} = \frac{7k}{15} = 7 \Rightarrow k = 15.$$

3. (14 pts) Consider the sequence $a_n = \dfrac{1 + \ln(n)}{n}$ for $n = 3, 4, 5, \ldots$. Be sure to justify your answers to the following questions.

   (a) Is the sequence monotonic?

   (b) Is the sequence bounded?

   (c) Does the sequence converge? If so, what does it converge to?

Solution:

(a) Yes, the sequence is monotone decreasing.
To justify this, set $f(x) = \dfrac{1 + \ln x}{x}$ and compute $f'(x) = -\dfrac{\ln x}{x^2}$. We see that the derivative is negative for all $x \geq 3$, hence $f(x)$ is decreasing. Since $f(n) = a_n$, the sequence is monotone decreasing.

(b) Since the sequence is monotone decreasing, it is bounded above by its first element $a_3 = \dfrac{1 + \ln(3)}{3}$. Since all of the terms are positive, the sequence is bounded below by 0.

(c) A monotone decreasing series converges by the Monotonic Sequence Theorem. To find the limit, we set $f(x) = \dfrac{1 + \ln x}{x}$ and consider

$$\lim_{x\to\infty} \frac{1 + \ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0$$

which follows from L'Hopital's Rule. Therefore, the sequence converges to 0.

4. (18 pts) Let the nth partial sum of a series $\sum_{n=1}^{\infty} a_n$ equal $s_n = 5 - \dfrac{5}{\sqrt{n+1}}$. Be sure to justify your answers to the following questions.

   (a) Find the values of a_1 and a_2, the first two terms of the series. Write each term as the sum or difference of two fractions.

   (b) Does $\sum_{n=1}^{\infty} a_n$ converge or diverge? If it converges, find its sum.

   (c) Does a_n converge or diverge? If it converges, what does it converge to?

   (d) Does $\sum_{n=1}^{\infty} s_n$ converge or diverge? If it converges, find its sum.

Solution:

(a)

$$\begin{aligned} a_1 &= s_1 = \boxed{5 - \frac{5}{\sqrt{2}}} \\ a_2 &= s_2 - s_1 = (5 - \frac{5}{\sqrt{3}}) - (5 - \frac{5}{\sqrt{2}}) = \boxed{-\frac{5}{\sqrt{3}} + \frac{5}{\sqrt{2}}} \end{aligned}$$

(b) We first note that $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \sum_{k=1}^{\infty} a_k$, if the limit of the partial sums exists. Since $\lim_{n\to\infty} \left[5 - \frac{5}{\sqrt{n+1}}\right] = 5$. we have $\sum_{n=1}^{\infty} a_n$ converges to 5.

(c) Since the series $\sum_{n=1}^{\infty} a_n$ converges, we must have the limit of the sequence equal to 0. That is, $\lim_{n\to\infty} a_n = 0$. (Note: If $\lim_{n\to\infty} a_n \neq 0$ then the series diverges by the Test for Divergence.)

(d) Since $\lim_{n\to\infty} s_n = 5$, by the Test for Divergence, we have $\sum_{n=1}^{\infty} s_n$ diverges.

5. (16 pts) Let b be a constant. Consider the geometric series given by $b - \frac{b^2}{4} + \frac{b^3}{16} - \cdots$.

(a) Write the series in the form $\sum_{n=1}^{\infty} ar^{n-1}$.

(b) Find all values of b for which the series will converge.

(c) If the sum of the series is $8/5$, what is the value of b?

Solution:

(a) $b - \frac{b^2}{4} + \frac{b^3}{16} - \cdots = \sum_{n=1}^{\infty} b\left(\frac{-b}{4}\right)^{n-1}$

(b) This is a geometric series with $a = b$ and $r = \frac{-b}{4}$. The series will converge if $-1 < r < 1$. Thus, we need $-4 < b < 4$.

(c) We require

$$\sum_{n=1}^{\infty} b\left(\frac{-b}{4}\right)^{n-1} = \frac{b}{1 + (b/4)} = \frac{4b}{4+b} = \frac{8}{5}$$

Solving $\frac{4b}{4+b} = \frac{8}{5}$ yields $\boxed{b = \frac{8}{3}}$