1. (34 pts) Evaluate the following integrals and simplify your answers.

   (a) $\int \left(\tan^2\theta + 1\right)\sec^2\theta \, d\theta$

   (b) $\int \dfrac{2x^2 - 5x + 6}{x^3 + 3x}\, dx$

   (c) $\int \dfrac{dx}{x^2\sqrt{25 - x^2}}$

   Solution:

   (a)

$$\begin{aligned}\int \left(\tan^2\theta + 1\right)\sec^2\theta \, d\theta &= \int \underbrace{\tan^2\theta\sec^2\theta}_{\substack{u=\tan\theta \\ du=\sec^2\theta\, d\theta}} d\theta + \int \sec^2\theta\, d\theta \\ &= \int u^2\, du + \int \sec^2\theta\, d\theta \\ &= \boxed{(1/3)\tan^3\theta + \tan\theta + C}\end{aligned}$$

   (b) Using partial fractions, we have

$$\frac{2x^2 - 5x + 6}{x^3 + 3x} = \frac{2x^2 - 5x + 6}{x(x^2+3)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 3}$$

   Solving yields $A = 2$, $B = 0$ and $C = -5$. The original integral becomes:

$$\begin{aligned}\int \frac{2x^2 - 5x + 6}{x^3 + 3x}\, dx &= \int \frac{2}{x}\, dx - \int \frac{5}{x^2+3}\, dx \\ &= \boxed{2\ln|x| - \frac{5}{\sqrt{3}}\tan^{-1}(x/\sqrt{3}) + C}\end{aligned}$$

   (c) For this integral, use trig substitution with $x = 5\sin\theta$ and $dx = 5\cos\theta\, d\theta$ and obtain

$$\begin{aligned}\int \frac{dx}{x^2\sqrt{25 - x^2}} &= \int \frac{5\cos\theta}{25\sin^2\theta\sqrt{25 - 25\sin^2\theta}}\, d\theta \\ &= \int \frac{1}{25\sin^2\theta}\, d\theta \\ &= \frac{1}{25}\int \csc^2\theta\, d\theta \\ &= \frac{-1}{25}\cot\theta + C \\ &= \boxed{-\frac{\sqrt{25 - x^2}}{25x} + C}\end{aligned}$$

   where the final equality comes from the reference triangle for $x = 5\sin\theta$.

2. (26 points) Consider the integral $I = \int_{-1}^{1} (2-x)e^x\,dx$.

(a) Estimate the value of I using the trapezoidal approximation T_2. Express your answer in terms of the number e and simplify.

(b) Estimate the error for the approximation T_2. Express your answer in terms of the number e and simplify.

(c) Find the exact value of the integral.

Solution: Let $f(x) = (2-x)e^x$.

(a)

$$\begin{aligned} T_2 &= \frac{\Delta x}{2}\left[f(-1) + 2f(0) + f(1)\right] \\ &= \frac{1}{2}\cdot\frac{1-(-1)}{2}\left[(2-(-1))e^{-1} + 2(2-0)e^0 + (2-1)e^1\right] \\ &= \boxed{\frac{3e^{-1}+4+e}{2}} \end{aligned}$$

(b)

$$|E_T| \leq \frac{K[1-(-1)]^3}{12n^2} = \frac{8K}{12\cdot 2^2} = \frac{K}{6}$$

$$\begin{aligned} f'(x) &= (2-x)e^x - e^x = (1-x)e^x \\ f''(x) &= (1-x)e^x - e^x = -xe^x \\ |f''(x)| &= |-xe^x| = |-x||e^x| = |x|e^x \end{aligned}$$

On the interval $[-1,1]$, $|x|$ attains its maximum value at $x=\pm 1$ and e^x attains its maximum value at $x=1$. Therefore, $|f''(x)| = |x|e^x$ attains its maximum value on $[-1,1]$ at $x=1$.

$$|f''(x)| \leq |f''(1)| = |-e| = e.$$

Therefore, let $K = e$, so that $\boxed{|E_T| \leq \frac{e}{6}}$.

(c)

$$I = \int_{-1}^{1} (2-x)e^x\,dx$$

Apply Integration by Parts, as follows:

$$u = 2-x \qquad dv = e^x\,dx$$

$$du = -dx \qquad v = e^x$$

$$\begin{aligned} I &= (2-x)e^x\Big|_{-1}^{1} + \int_{-1}^{1} e^x\,dx \\ &= (2-x)e^x\Big|_{-1}^{1} + e^x\Big|_{-1}^{1} \\ &= e - 3e^{-1} + e - e^{-1} \\ &= \boxed{2e - 4e^{-1}} \end{aligned}$$

3. (22 points) Determine whether the following integrals are convergent or divergent. Explain your reasoning fully for each integral. **If the integral converges, find its value**.

(a) $\displaystyle\int_2^\infty \frac{x^2}{\sqrt{x^6-4}}\,dx$

(b) $\displaystyle\int_{-1}^0 \frac{e^{1/x}}{x^2}\,dx$

Solution:

(a) For $x \geq 2$ we have

$$0 \leq x^6 - 4 \leq x^6$$

so

$$0 \leq (x^6-4)^{1/2} \leq x^3$$

Thus,

$$0 \leq \frac{1}{x} = \frac{x^2}{x^3} \leq \frac{x^2}{(x^6-4)^{1/2}}$$

Since we know

$$\int_2^\infty \frac{1}{x}\,dx = \lim_{t\to\infty}\int_2^t \frac{1}{x}\,dx = \lim_{t\to\infty} \ln(x)\big|_2^t = \lim_{t\to\infty}(\ln t - \ln 2) \text{ diverges}$$

then the original integral diverges by the Comparison Test for Integrals.

Alternate Solution: Let $x^3 = 2\sec\theta$. Then $3x^2\,dx = 2\sec\theta\tan\theta\,d\theta$.

$$\begin{aligned}\int_2^\infty \frac{x^2}{\sqrt{x^6-4}}\,dx &= \lim_{a\to\pi/2^-}\int_{\text{arcsec}\,4}^a \frac{\frac{2}{3}\sec\theta\tan\theta}{2\tan\theta}d\theta \\ &= \lim_{a\to\pi/2^-}\int_{\text{arcsec}\,4}^a \frac{1}{3}\sec\theta\,d\theta \\ &= \lim_{a\to\pi/2^-}\frac{1}{3}\Big[\ln|\sec\theta+\tan\theta|\Big]_{\text{arcsec}\,4}^a \\ &= \lim_{a\to\pi/2^-}\frac{1}{3}\left[\ln|\sec a+\tan a| - \ln\left|4+\sqrt{15}\right|\right] = \infty\end{aligned}$$

The given integral is $\boxed{\text{divergent}}$.

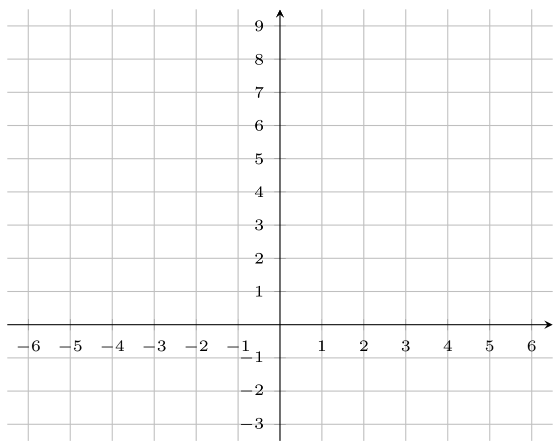
(b)

$$\int_{-1}^0 \frac{e^{1/x}}{x^2}\,dx = \lim_{a\to 0^-}\int_{-1}^a \underbrace{\frac{e^{1/x}}{x^2}}_{\substack{u=1/x \\ du=-(1/x^2)\,dx}}\,dx$$

$$= \lim_{a\to 0^-} \int_{-1}^{1/a} -e^u \, du$$
$$= \lim_{a\to 0^-} -e^u\big|_{-1}^{1/a}$$
$$= \lim_{a\to 0^-} (-e^{1/a} + e^{-1}) = \boxed{e^{-1}}$$

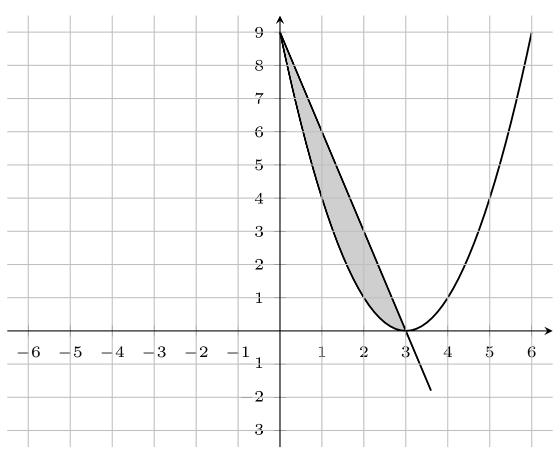
Thus, the improper integral converges by direct evaluation.

4. (18 points) Consider the region $\mathcal{R}$ in Quadrant 1 bounded by the curves $y = (x-3)^2$ and $y = -3x+9$.
   (a) Use the grid provided below to sketch the region $\mathcal{R}$. Shade in the region $\mathcal{R}$.
   (b) Using only disks or washers, set up, but do not evaluate, an integral to find the volume of the solid generated by rotating $\mathcal{R}$ about:
      i. the x-axis,
      ii. the line $x = 3$.



Solution:

(a)



(b) i. Volume for rotating around the x-axis is given by

$$\boxed{V = \pi \int_0^3 \left[(-3x+9)^2 - (x-3)^4\right] \, dx}$$

ii. To find the volume for rotating around the line $x = 3$ using washers, we need to use $y = (x-3)^2$ to find x as a function of y. We obtain $x = -y^{1/2} + 3$. Likewise, $y = -3x + 9$ becomes $x = (9 - y)/3$. Then, the requested volume is

$$\begin{aligned} V &= \pi \int_0^9 \left[(3 - (3 - y^{1/2}))^2 - \left(3 - \frac{9-y}{3}\right)^2 \right] dy \\ &= \boxed{\pi \int_0^9 \left[y - \left(3 - \frac{9-y}{3}\right)^2 \right] dy} \end{aligned}$$