1. (24 points) Evaluate each of the following:

$$(a) \quad \int \cos^5 x\,dx \qquad\qquad (b) \quad \int_1^\infty \frac{\arctan(1/x)}{x^2}\,dx$$

Solution:

(a) We will make use of a Pythagorean identity and the substitution $y = \sin x$, $du = \cos x\,dx$:

$$\begin{aligned}\int \cos^5 x\,dx &= \int (\cos^2 x)^2 \cos x\,dx \\ &= \int (1-\sin^2 x)^2 \cos x\,dx \\ &= \int (1-u^2)^2\,du \\ &= \int 1 - 2u^2 + u^4\,du \\ &= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C \\ &= \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C.\end{aligned}$$

(b) We will first rewrite the improper integral as the limit of a definite integral:

$$\int_1^\infty \frac{\arctan(1/x)}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{\arctan(1/x)}{x^2}\,dx.$$

Next, we will apply the substitution $w = \frac{1}{x}$. Note that $dw = -\frac{1}{x^2}dx$, the new lower limit of integration is $w = 1$, and the new upper limit of integration is $w = \frac{1}{t}$:

$$\begin{aligned}\lim_{t\to\infty}\int_1^t \frac{\arctan(1/x)}{x^2}\,dx &= \lim_{t\to\infty} -\int_1^{\frac{1}{t}} \arctan(w)\,dw \\ &= \lim_{t\to\infty}\int_{\frac{1}{t}}^1 \arctan(w)\,dw.\end{aligned}$$

Now, we will use integration by parts where $u = \arctan(w)$ and $dv = dw$. It follows from this that $du = \frac{1}{1+w^2}dw$ and $v = w$:

$$\begin{aligned}\lim_{t\to\infty}\int_{\frac{1}{t}}^1 \arctan(w)\,dw &= \lim_{t\to\infty} w\arctan(w)|_{\frac{1}{t}}^1 - \int_{\frac{1}{t}}^1 \frac{w}{1+w^2}dw \\ &= \lim_{t\to\infty}\left(\frac{\pi}{4} - \frac{\arctan(1/t)}{t}\right) - \frac{1}{2}\left(\ln 2 - \ln(1+1/t^2)\right) \\ &= \frac{\pi}{4} - \frac{1}{2}\ln 2.\end{aligned}$$

Alternatively, when antidifferentiating, one can choose not to use a substitution and instead use integration by parts with $u = \arctan(1/x)$ and $dv = \frac{1}{x^2}\,dx$. This will yield an antiderivative of $-\frac{1}{x}\arctan(x) - \ln|x| + \frac{1}{2}\ln(1+x^2) + C$ which will lead to the same final answer of $\frac{\pi}{4} - \frac{1}{2}\ln 2$.

2. (28 points) Consider the integral $\displaystyle\int_0^1 e^{-x^2}\,dx$. All parts of this problem refer to this integral.

   (a) Write out an approximation for this integral using the midpoint rule with $n = 4$. (You DO NOT need to simplify your final answer. Your answer should be in a form that *could* be directly input into a calculator.)

   (b) Using the techniques from this course, what is the error bound of the approximation from (a)?

   (c) Use the MacLaurin series for e^x to find a series representation for this integral.

   (d) Use the Alternating Series Estimation Theorem to determine the error bound if the first four nonzero terms of the series from (c) are used to approximate the value of the integral. (Be sure to verify the conditions of the Alternating Series Estimation Theorem hold true in this example.)

Solution:

(a) If we let $f(x) = e^{-x^2}$, then we have

$$\begin{aligned}\int_0^1 e^{-x^2}\,dx &\approx [f(1/8) + f(3/8) + f(5/8) + f(7/8)]\,(1/4)\\ &= \left[e^{-1/64} + e^{-9/64} + e^{-25/64} + e^{-49/64}\right](1/4).\end{aligned}$$

(b) Recall that the error bound for the midpoint rule is $|E_M| \leq \dfrac{K(b-a)^3}{24n^2}$. We know that $b = 1$, $a = 0$, $n = 4$, and that K is an upper bound on the second deritvative of the integrand over $[0,1]$. We see that $f''(x) = (4x^2 - 2)e^{-x^2}$, so we have

$$|f''(x)| = |(4x^2 - 2)||e^{-x^2}| < 2 \cdot e^0 = 2$$

because $4x^2 - 2$ is increasing on $[0,1]$ and takes values of -2 and 2 on $x = 0$ and $x = 1$, respectively, and e^{-x^2} is decreasing on $[0,1]$ and takes values of 1 and $\frac{1}{e}$ on $x = 0$ and $x = 1$, respectively. So, we shall use $K = 2$. Thus, the error for the approximation in (a) is

$$|E_M| \leq \frac{K(b-a)^3}{24n^2} \leq \frac{2(1-0)^3}{24 \cdot 4^2} = \frac{1}{192}.$$

(c) Recall that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

So,

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}.$$

Thus,

$$\begin{aligned}\int_0^1 e^{-x^2}\,dx &= \int_0^1 \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}\right] dx\\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)}.\end{aligned}$$

(d) We note that $b_n = \frac{1}{n!(2n+1)} > 0$ is decreasing because

$$b_{n+1} = \frac{1}{(n+1)!(2n+3)} < \frac{1}{n!(2n+1)} = b_n$$

and

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n!(2n+1)} = 0.$$

So, the Alternating Series Estimation Theorem does apply.
Since we see that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \cdots,$$

then we know that the estimation

$$\int_0^1 e^{-x^2}\,dx \approx 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42}$$

would have an error of at most $\frac{1}{216}$.

3. (24 points) A gloopy is an intelligent species that lives on the planet Blorpy. Gloopies enjoy getting around the planet Blorpy with jet packs. This one particular gloopy takes off at $t = 0$, travels in the air above a straight street, and then lands back down on the street. Specifically, after t seconds, they are $x(t) = t^2 + 10t$ feet down the street with a height of $y(t) = 100t - 4t^2$ feet.

   (a) At what time does the gloopy land back on the street?

   (b) Set up but do not evalaute an integral for the length of the path that the gloopy traveled through the air between $t = 0$ seconds and the time found in (a).

   (c) Set up but do not evalaute an integral for the area between the gloopy's path and the street between $t = 0$ and the time found in (a).

Solution:

(a) We need to solve $y(t) = 0$:

$$\begin{aligned} 100t - 4t^2 &= 0 \\ 4t(25 - t) &= 0 \\ t &= 0, 25. \end{aligned}$$

So, the gloopy lands back on the street after $t = 25$ seconds. (Note that $t = 0$ is the time when the gloopy lifts off.)

(b) We wish the find the arc length of the path that the gloopy travels.

$$\begin{aligned} L &= \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2} dt \\ &= \int_0^{25} \sqrt{[2t + 10]^2 + [100 - 8t]^2} dt. \end{aligned}$$

(c) The area under the parameterized curve is given by

$$\begin{aligned} A &= \int_a^b y\,dx \\ &= \int_0^{25} (100t - 4t^2)(2t+10)\,dt. \end{aligned}$$

4. (14 points) Find the sum of $\displaystyle\sum_{n=2}^{\infty} \frac{2}{n^2-1}$.

   Solution:

   We begin by finding the partial fraction decomposition of the nth term:

$$\frac{2}{n^2-1} = \frac{A}{n-1} + \frac{B}{n+1} = \frac{A(n+1)+B(n-1)}{n^2-1}.$$

   From this, we find $A = 1$ and $B = -1$.

   Since the series is telescoping, we use this partial fraction decomposition to find a simplified form of the sequence of partial sums:

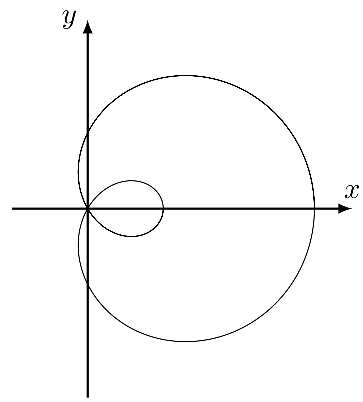
$$\begin{aligned} s_n &= \sum_{i=2}^{n} \frac{2}{i^2-1} \\ &= \sum_{i=2}^{n} \left[\frac{1}{i-1} - \frac{1}{i+1}\right] \\ &= \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \cdots \\ &\cdots + \left(\frac{1}{n-4} - \frac{1}{n-2}\right) + \left(\frac{1}{n-3} - \frac{1}{n-1}\right) + \left(\frac{1}{n-2} - \frac{1}{n}\right) + \left(\frac{1}{n-1} - \frac{1}{n+1}\right) \\ &= 1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}. \end{aligned}$$

   From this, we see

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{2}{n^2-1} &= \lim_{n\to\infty} s_n \\ &= \lim_{n\to\infty} \left[1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}\right] \\ &= \frac{3}{2}. \end{aligned}$$

5. (24 points) Consider the curve given by the following Polar equation which is plotted below:

$$r = 1 + 2\cos\theta, \qquad 0 \le \theta \le 2\pi$$



(a) Find the equation of the tangent line when $\theta = \frac{\pi}{3}$.

(b) Set up but do not evaluate an integral that equals the length of the inner loop.

Solution:

(a) When $\theta = \frac{\pi}{3}$, we have $r = 2$. So, the Cartesian coordinates for the point of tangency are $(1, \sqrt{3})$.
Next, we find $\frac{dy}{dx}$ when $\theta = \frac{\pi}{3}$:

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \\ &= \frac{\frac{d}{d\theta}((1+2\cos\theta)\sin\theta)}{\frac{d}{d\theta}((1+2\cos\theta)\cos\theta)} \\ &= -\frac{\cos\theta + 2\cos(2\theta)}{\sin\theta + 4\sin\theta\cos\theta}.\end{aligned}$$

When $\theta = \frac{\pi}{3}$, this derivative evaluates to $\frac{\sqrt{3}}{9}$.
This point and slope yields the line $y = \frac{\sqrt{3}}{9}(x+8)$.

(b) To determine the limits of integration, we need to know the values of θ when $r = 0$. Solving $1 + 2\cos\theta = 0$ gives solutions $\theta = \frac{2\pi}{3}$ and $\frac{4\pi}{3}$.

$$\begin{aligned}L &= \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \\ &= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \sqrt{(1+2\cos\theta)^2 + (-2\sin\theta)^2}\, d\theta \\ &= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \sqrt{5 + 4\cos\theta}\, d\theta.\end{aligned}$$

6. (8 points) Consider the region bounded by $y = \sqrt{x}$, $y = 2$, and $x = 0$. Set up but do not evaluate an integral expression for the volume of the solid when this region is rotated about the line $y = -1$.

Solution: Using washers, we have radii of $R = 2 + 1 = 3$ and $r = \sqrt{x} + 1$. So, the volume would be

$$V = \pi \int_0^4 3^2 - (\sqrt{x}+1)^2\, dx.$$

If instead we use shells, we will be integrating with respect to y with a radius of $r = y + 1$ and a height of $h = y^2$. So, the volume would be

$$V = 2\pi \int_0^2 (y+1)y^2\,dy.$$

7. (8 points) Consider the following conic section:

$$x^2 + 2y^2 - 6x + 5 = 0.$$

   (a) Determine if this conic section is a **parabola**, an **ellipse**, or a **hyperbola**.

   (b) If it is an ellipse or a hyperbola, determine its center. If it is a parabola, determine its vertex.

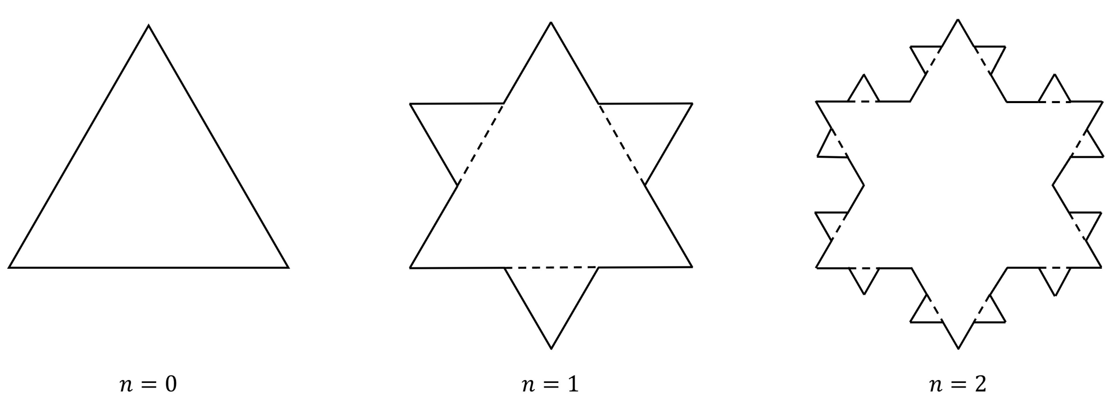
Solution: We will manipulate the equation and complete the square to see which standard equation type we end up with:

$$\begin{aligned} x^2 + 2y^2 - 6x + 5 &= 0 \\ (x^2 - 6x) + 2y^2 &= -5 \\ (x^2 - 6x + 9) + 2y^2 &= -5 + 9 \\ (x-3)^2 + 2y^2 &= 4 \\ \frac{(x-3)^2}{4} + \frac{y^2}{2} &= 1. \end{aligned}$$

We see that this equation represents an **Ellipse** whose center is $(3, 0)$.

8. (20 points) Consider the fractal that is constructed iteratively as follows:

Begin with an equilateral triangle whose side lengths all equal 1. Each iteration of the construction process involves dividing each line segment from the previous iteration's perimeter into thirds. A new, smaller equilateral triangle is then built on the middle third of each such line segment, thereby increasing the area of the shape and the number of line segments comprising the perimeter. The initial triangle and the results of the first two iterations are depicted here:



   (a) Let the sequence $\{a_n\}_{n=0}^{\infty}$ represent the total number of line segments on the perimeter of the shape after n iterations. Find a_n. (You should be counting the solid line segments, not the dashed line segments.)

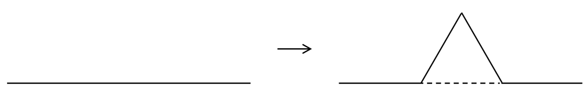
   (b) Let the sequence $\{b_n\}_{n=0}^{\infty}$ represent the length of each line segment on the perimeter of the shape after n iterations. Find b_n.

(c) Let the sequence $\{c_n\}_{n=1}^{\infty}$ represent the number of new triangles that are added to the shape during the n^{th} iteration. Find c_n.

(d) If the construction process is continued indefinitely, what would be the limiting area of the shape?
(*Note:* The area of an equilateral triangle with side length L is $\sqrt{3}\,L^2/4$.)

Solution:

(a) Each iteration involves replacing each line segment on the perimeter with four smaller segments:



That is, at each iteration, one line segment is turned into four distinct line segments. Since there are initially three segments, $\boxed{a_n = (3)(4^n)}$

(b) By construction, the length of each perimeter line segment after any interation is $1/3$ the length of each perimeter line segment before that iteration. Since the initial segment length is 1, $\boxed{b_n = (1/3)^n}$

(c) During each iteration, one triangle is added for each perimeter line segment that was present at the end of the previous iteration. Therefore, $c_n = a_{n-1} = \boxed{(3)(4^{n-1})}$

(d) $$\begin{aligned} A = \text{Limiting area} &= \text{area of original triangle} \\ &+ \sum_{n=1}^{\infty} (\text{ number of new triangles added during iteration } n)(\text{area of each new triangle}) \end{aligned}$$

$$A = \frac{\sqrt{3}}{4} + \sum_{n=1}^{\infty} c_n \left(\frac{\sqrt{3}\,{b_n}^2}{4} \right) = \frac{\sqrt{3}}{4} \left[1 + \sum_{n=1}^{\infty} c_n\,{b_n}^2 \right]$$

$$c_n\,{b_n}^2 = [(3)(4^{n-1})] \left[\left(\frac{1}{3} \right)^n \right]^2 = (3)(4^{n-1}) \left[\left(\frac{1}{3} \right)^2 \right]^n = (3)(4^{n-1}) \left(\frac{1}{9} \right)^n = \frac{1}{3} \cdot \left(\frac{4}{9} \right)^{n-1}$$

$$A = \frac{\sqrt{3}}{4} \left[1 + \frac{1}{3} \sum_{n=1}^{\infty} \left(\frac{4}{9} \right)^{n-1} \right]$$

Since $|4/9| < 1$, the series in the preceding equation is a convergent geometric series. Therefore,

$$A = \frac{\sqrt{3}}{4} \left[1 + \frac{1}{3} \left(\frac{1}{1 - 4/9} \right) \right] = \frac{\sqrt{3}}{4} \left[1 + \frac{1}{3} \cdot \frac{9}{5} \right] = \frac{\sqrt{3}}{4} \cdot \frac{8}{5} = \boxed{\frac{2\sqrt{3}}{5}}$$